

# QUANTUM INVARIANTS OF 3-MANIFOLDS VIA LINK SURGERY PRESENTATIONS AND NON-SEMI-SIMPLE CATEGORIES

FRANCESCO COSTANTINO, NATHAN GEER, AND BERTRAND PATUREAU-MIRAND

**ABSTRACT.** In this paper we construct invariants of 3-manifolds “à la Reshetikhin-Turaev” in the setting of non-semi-simple ribbon tensor categories. We give concrete examples of such categories which lead to a family of 3-manifold invariants indexed by the integers. We prove this family of invariants has several notable features, including: they can be computed via a set of axioms, they distinguish homotopically equivalent manifolds that the standard Reshetikhin-Turaev-Witten invariants do not, and they allow the statement of a version of the Volume Conjecture and a proof of this conjecture for an infinite class of links.

## INTRODUCTION

The Reshetikhin-Turaev-Witten 3-manifold invariants are extremely important and intriguing objects in low-dimensional topology. These invariants can be computed combinatorially and lead to Topological Quantum Field Theories (TQFTs) and representations of mapping class groups. They have been studied extensively but still have a mysterious topological significance. Reshetikhin and Turaev [33] gave the first rigorous construction of these invariants which have become known as quantum invariants of 3-manifolds. Their proof uses surgery to reduce the general case to the case of links in  $S^3$  then applies certain quantum invariants of links associated to quantum  $\mathfrak{sl}(2)$  at a root of unity (see [32]). The reduction of the topology of 3-manifolds to the theory of links in  $S^3$  is well-known: any closed orientable connected 3-manifold is obtained by surgery on some framed link in  $S^3$ . Two manifolds  $M_L$  and  $M_{L'}$  obtained by surgery on  $L$  and  $L'$ , respectively, are homeomorphic if and only if the framed links  $L$  and  $L'$  may be related by a series of Kirby moves (see [24]).

Roughly speaking, the construction of the quantum invariant of 3-manifolds defined by Reshetikhin and Turaev can be described as follows (for more details see [33]). Consider the quotient  $\overline{U}$  of  $U_q(\mathfrak{sl}(2))$  defined by setting  $E^r = F^r = 0$  and  $K^r = 1$ , where  $q$  is a root of unity of order  $2r$  and  $E, F$  and  $K$  are the generators of  $U_q(\mathfrak{sl}(2))$ . Any finite dimensional  $\overline{U}$ -module  $V$  decomposes as  $V \cong \oplus_{i=1}^n V_i \oplus W$  where  $V_i$  is a simple  $\overline{U}$ -module with non-zero quantum dimension and  $W$  is a  $\overline{U}$ -module with zero quantum dimension. By quotienting the category of finite dimensional  $\overline{U}$ -modules by  $\overline{U}$ -modules with zero quantum dimension one obtains a modular category  $\mathcal{D}$ . Loosely speaking, a modular category is a semi-simple ribbon category with a finite number of isomorphism classes of simple objects satisfying some axioms. Let  $M$  be a manifold obtained by surgery on  $L$ . If the  $i^{\text{th}}$  component of  $L$  is labeled by a simple module  $V_i$  of  $\mathcal{D}$  then consider the weighted link invariant

$$(1) \quad \left( \prod_i \text{qdim}_{\mathcal{D}}(V_i) \right) F(L),$$

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where  $\text{qdim}_{\mathcal{D}}$  is the quantum dimension and  $F$  is the Reshetikhin-Turaev link invariant associated to  $\mathcal{D}$ , see [32]. The invariant of  $M$  is the finite sum of such weighted link invariants over all possible labelings of  $L$ . The Kirby moves correspond to certain algebraic identities. Using the semi-simplicity of  $\mathcal{D}$  it can be shown that the weight sum is preserved by the Kirby moves. Thus, the existence of this sum depends heavily on the fact that  $\mathcal{D}$  contains only a finite number of isomorphism classes of simple objects all having non-zero quantum dimensions.

Most of the research on quantum invariants is based on semi-simple (modular) categories. Apart from a few exceptions ([19, 25, 35]), the work that has been done on quantum invariants of links and 3-manifolds coming from non-semi-simple representation theory has been centered on examples related to quantum  $\mathfrak{sl}(2)$ . Such work has been initiated by the independent and seminal works of Akutsu, Deguchi, Ohtsuki [2], Kashaev [21], Viro [36] and others. In [28], Murakami and Murakami showed that the Akutsu-Decuchi-Ohtsuki (ADO) invariants are related to Kashev's invariants. The ADO invariants are also related to the multivariable Alexander polynomials (see [26]). More recently, Murakami and Nagatomo [29] defined "logarithmic invariants," Benedetti and Baseilhac [3] extended Kashaev's construction to a "quantum hyperbolic field theory," Kashaev and Reshetikhin [23] defined tangle invariants from non semi-simple categories and Andersen and Kashaev [1] constructed a TQFT out of quantum Teichmüller theory.

This body of work is deep and requires new techniques involving algebra, topology, geometry and mathematical physics. Moreover, these invariants are related to well known problems, including the Volume Conjecture (see [21, 28]). However, to our knowledge no one has constructed a quantum 3-manifold invariant based on link surgery presentations arising from the non-semi-simple categories of representations of quantum groups. The purpose of this paper is to fill this gap by giving a general construction of such invariants. This approach is useful and powerful for several reasons, including: 1) it produces new invariants which are computable and distinguish homotopically equivalent manifolds that the standard Reshetikhin-Turaev-Witten invariants do not (see Subsections 2.1 and 2.3), 2) it allows the statement of a version of the Volume Conjecture and a proof of this conjecture for the so called fundamental hyperbolic links (see Subsection 2.2), 3) our construction applied to modular categories arising from simple Lie algebras yields invariants which are equal to the usual Reshetikhin-Turaev-Witten invariants (see Remark 3.13), and 4) it produces new invariants from non-semi-simple representation theory associated to any quantum simple Lie algebra (see Section 6).

We will now describe the results of this paper in the context of quantum  $\mathfrak{sl}(2)$ . Let  $U_q^H \mathfrak{sl}(2)$  be the quantization of  $\mathfrak{sl}(2)$  defined in Subsection 5.1. This quantization has five generators:  $E, F, H, K, K^{-1}$  where  $H$  can be considered as the logarithm of  $K$ . Let  $\bar{U}_q^H \mathfrak{sl}(2)$  be  $U_q^H \mathfrak{sl}(2)$  modulo the relations  $E^r = F^r = 0$ . We consider the category  $\mathcal{C}$  of finite dimensional  $H$ -weight modules over  $\bar{U}_q^H \mathfrak{sl}(2)$  such that  $K$  acts as the operator  $q^H$ . Unlike the modular category  $\mathcal{D}$  discussed above we do not require that  $K^r = 1$  instead  $K^r$  acts as a scalar on a simple module of  $\mathcal{C}$ . Also unlike  $\mathcal{D}$  here we do not take a quotient of the category of  $\bar{U}_q^H \mathfrak{sl}(2)$ -modules. The category  $\mathcal{C}$  is a ribbon category which is not semi-simple and has an infinite number of isomorphism classes of simple modules indexed by  $\mathbb{C}$ . Every module indexed by  $(\mathbb{C} \setminus \mathbb{Z}) \cup \{kr : k \in \mathbb{Z}\} \subset \mathbb{C}$  has a vanishing quantum dimension. It is these modules that we use to define our invariant. Note that the module corresponding to  $0 \in \mathbb{C}$  is Kashaev's module used in Murakami and Murakami's reformulation of the Volume Conjecture [28].

The invariants of this paper are defined by modifying the usual construction in three main steps. First, we replace vanishing functions  $\text{qdim}$  and  $F$  in Equation (1) with the modified dimension  $\mathbf{d}$  and re-normalized quantum invariant  $F'$  associated to  $\mathcal{C}$ , defined in [15]. Second, we introducing a trivalent graph  $T$  in a closed orientable connected 3-manifold  $M$ . The usual weighted sum (described above) applied to  $\mathcal{C}$  is infinite because the isomorphism classes of simple

modules in  $\mathcal{C}$  are indexed by the infinite set of complex numbers  $\mathbb{C}$ . To overcome this problem, a finite number of modules are selected using a cohomology class in  $H^1(M \setminus T; \mathbb{C}/2\mathbb{Z})$ . In particular, if  $M$  is obtained by surgery on  $L$ , then the cohomology class is used to obtain a coloring of the components of  $L$  by simple modules, indexed by complex numbers. Certain integral translates of such a coloring lead to a finite sum of weighted link invariants as suggested above. The final step is to show that this weighted sum is preserved by the Kirby moves. As mentioned above, the standard proof uses that the category is semi-simple. To deal with this obstruction one can recognize that under certain admissibility conditions the tensor product of two simple modules in  $\mathcal{C}$  is semi-simple. The result is a topological invariant  $\mathbf{N}$  of triples (a closed oriented 3-manifold  $M$ , a trivalent graph  $T$  in  $M$ , an element  $\omega$  of  $H^1(M \setminus T; \mathbb{C}/2\mathbb{Z})$ ).

When a surgery presentation of a triple  $(M, T, \omega)$  leads to the weighted sum discussed above we call it *computable*. Sometimes a triple may not admit a computable presentation but a suitable *H-stabilization* (a connected sum of the Hopf link with a component of  $T$ ) may; so we extend our definition of the invariant by first *H-stabilizing* then re-normalizing the invariant of the new triple to account for such a stabilization. To deal with the extreme cases when no such *H-stabilization* exists (i.e. when  $T$  is empty and  $\omega$  comes from an integral cohomology class in  $H^1(M; \mathbb{Z}/2\mathbb{Z})$ ) we define another invariant  $\mathbf{N}^0(M, T, \omega) \in \mathbb{C}$  which naturally completes the definition of  $\mathbf{N}(M, T, \omega)$  in the following sense:  $\mathbf{N}^0$  is defined for any triple but it is zero if a computable surgery presentation exists (see Theorem 1.8). In Subsection 2.4 we prove that  $\mathbf{N}^0(P, \emptyset, 0) \neq \mathbf{N}^0(S^3, \emptyset, 0) = 1$  where  $P$  is the Poincaré sphere,  $S^3$  is the 3-sphere and 0 is the trivial cohomology class. Thus,  $\mathbf{N}^0$  is non-trivial.

As the description of the construction above indicates, when using non-semi-simple categories to define invariants of links and 3-manifolds one needs additional structures. These structures are far from being obvious and highlight a striking interplay between topology, geometry and quantum algebra. Similar structures have been used in the work of [3, 15, 22, 23].

The work of this paper leads to a number of interesting open problems. First, it is natural to expect that our invariants extend to certain kinds of Topological Quantum Field Theories as the R-T-W invariants extend to the TQFTs first constructed in [5]. If such a TQFT exists then it should give rise to a new set of quantum representations of the Mapping Class Groups of surfaces. Second, as shown in Section 2, the invariants may be used to prove a version of the Volume Conjecture for certain pairs  $(M, T)$  where  $T$  is a link in a manifold  $M$ . Thus, it is natural to ask if this conjecture is true for other pairs  $(M, T)$ , in particular, in the case when  $T$  is empty. Finally, a natural problem is to relate the invariants of this paper to those defined in [15]; an analogous relationship exists between the usual R-T-W invariants and Turaev-Viro invariants. Giving such a relationship would be interesting also because it would relate the invariants of this paper with the generalized Kashaev invariants of [10], via the results of [14].

The paper is organized as follows. To make the paper accessible to more readers, in Section 1 we give an example of our invariants presented in a purely combinatorial way. This allows the reader to be able to compute and use our invariants immediately. Then in Section 2 we give several computations and properties of these invariants. In Section 3 we introduce a general categorical construction and state the main results of the paper. Section 4 contains the proofs of the results stated in Section 3. In Section 5 we prove that the combinatorial invariants of Section 1 are examples of the general theory defined in Section 3. Section 6 is devoted to showing that all quantized simple Lie algebras lead to examples of the general theory of this paper.

## 1. THE INVARIANTS $\mathbf{N}$ AND $\mathbf{N}^0$

**1.1. Notation.** All manifolds in the present paper are oriented, connected and compact unless explicitly stated. By a graph we always mean a finite graph with oriented edges (we allow loops

and multiple edges with the same vertices). Given a set  $Y$ , a graph is said to be  $Y$ -colored if it is equipped with a map from the set of its edges to  $Y$ . A *framed* graph  $\Gamma$  in an oriented manifold  $M$  is an embedding of  $\Gamma$  into  $M$  together with a vector field on  $\Gamma$  which is nowhere tangent to  $\Gamma$ , called the *framing* and so that at each  $v$  vertex of  $\Gamma$  the tangent vectors to the edges containing  $v$  belong to a same 2-dimensional tangent plane and are all distinct (this in particular induces a cyclic ordering of the incoming edges around  $v$ ). We assume that all edges of a vertex are tangent to the same plane which does not contain the vector of the framing. The framing is seen up to homotopy of vector fields constantly transverse to these tangent planes and so, together with the orientation of the manifold, gives a cyclic ordering of the edges of any vertex.

Let  $r$  be an integer greater or equal to 2 and let  $q = e^{i\pi/r}$ . For  $x \in \mathbb{C}$ , we use the notation  $q^x$  for  $e^{xi\pi/r}$  and let  $\{x\} = q^x - q^{-x}$ . For all  $x, y \in \mathbb{C}$  with  $x - y \in \{0, 1, \dots, r-1\}$ , let  $\begin{bmatrix} x \\ y \end{bmatrix} = \prod_{j=1}^{x-y} \frac{\{x+1-j\}}{\{j\}}$ . Set  $X_r = \mathbb{Z} \setminus r\mathbb{Z} \subset \mathbb{C}$  and define the *modified dimension*  $d : \mathbb{C} \setminus X_r \rightarrow \mathbb{C}$  by

$$(2) \quad d(\alpha) = (-1)^{r-1} \begin{bmatrix} \alpha + r - 1 \\ \alpha \end{bmatrix}^{-1}.$$

Finally, let  $H_r = \{1-r, 3-r, \dots, r-3, r-1\}$  and for any  $x \in \mathbb{C}$  let  $\bar{x} \in \mathbb{C}/2\mathbb{Z}$  be the class of  $x + r - 1$  modulo  $2\mathbb{Z}$  (so if  $r$  is odd,  $\bar{x}$  is the class of  $x$ ).

**1.2. Axiomatic definition of the invariant  $N$  of graphs in  $S^3$ .** In [2], Akutsu, Deguchi and Ohtsuki define generalized multivariable Alexander invariants, which contains Kashaev's invariants (see [22, 28]). In [26], Jun Murakami gives a framed version of these link invariants using the universal  $R$ -matrix of quantum  $\mathfrak{sl}(2)$  and calls them the colored Alexander invariant. Here we consider a generalization of these invariants (when  $r = 2$ , such a generalization was already considered by Viro in [36]).

As above, let  $r \in \mathbb{Z}$  with  $r \geq 2$  and let  $q = e^{i\pi/r}$ . Let  $\mathcal{G}$  be the set of oriented trivalent framed graphs in  $S^3$  whose edges are colored by element of  $\mathbb{C} \setminus X_r$ . In Subsection 5.2 we will show that the following axioms define an invariant  $N : \mathcal{G} \rightarrow \mathbb{C}$ . Let  $T, T' \in \mathcal{G}$ .

- (1) Let  $e$  be an edge of  $T$  colored by  $\alpha$ . If  $T'$  is obtained from  $T$  by changing the orientation of  $e$  and its color to  $-\alpha$  then  $N(T') = N(T)$ . In other words,

$$(N \text{ a}) \quad N \left( \begin{array}{c} | \\ \alpha \downarrow \\ | \end{array} \right) = N \left( \begin{array}{c} | \\ -\alpha \uparrow \\ | \end{array} \right).$$

- (2) Let  $\alpha, \beta, \gamma$  be the colors of a vertex  $v$  of  $T$ . If all the orientation of edges of  $v$  are incoming and  $\alpha + \beta + \gamma$  is not in  $H_r$  then  $N(T) = 0$ , i.e.

$$(N \text{ b}) \quad N \left( \begin{array}{c} \alpha \swarrow \quad \beta \searrow \\ \bullet \\ \gamma \uparrow \end{array} \right) = 0 \text{ if } \alpha + \beta + \gamma \notin H_r.$$

- (3) If  $T''$  denotes the connected sum of  $T$  and  $T'$  along an edge colored by  $\alpha$  then  $N(T'') = d(\alpha)^{-1} N(T) N(T')$ :

$$(N \text{ c}) \quad N \left( \begin{array}{c} \boxed{T} \xrightarrow{\alpha} \boxed{T'} \\ \boxed{\phantom{T}} \xleftarrow{\beta} \boxed{\phantom{T'}} \end{array} \right) = \delta_\alpha^\beta d(\alpha)^{-1} N \left( \begin{array}{c} \boxed{T} \end{array} \right) N \left( \begin{array}{c} \boxed{T'} \end{array} \right).$$

- (4) The invariant  $N$  has the following normalizations:

$$(N \text{ d}) \quad N \left( \begin{array}{c} \alpha \circlearrowleft \end{array} \right) = d(\alpha), \quad N \left( \begin{array}{c} \bullet \rightarrow \bullet \\ \bullet \leftarrow \bullet \end{array} \right) = 1, \quad N \left( \begin{array}{c} \bullet \downarrow \end{array} \right) = 0,$$

here we assume that the “ $\Theta$ ” graph is colored with any coloring which is not as in (N b).

- (5) If  $T''$  denotes the connected sum of  $T$  and  $T'$  along a vertex with compatible incident colored edges then  $N(T'') = N(T)N(T')$ :

$$(N \text{ e}) \quad N \left( \begin{array}{|c|c|} \hline T & T' \\ \hline \end{array} \right) = N \left( \begin{array}{|c|} \hline T \\ \hline \end{array} \right) N \left( \begin{array}{|c|} \hline T' \\ \hline \end{array} \right).$$

- (6)  $N$  is zero on split graphs:  $N(T \sqcup T') = 0$ .

- (7) The following relations hold whenever all appearing colors are in  $\mathbb{C} \setminus X_r$ :

$$(N \text{ f}) \quad N \left( \begin{array}{|c|} \hline \text{loop} \\ \hline \end{array} \right) = q^{\frac{\alpha^2 - (r-1)^2}{2}} N \left( \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \right),$$

$$(N \text{ g}) \quad N \left( \begin{array}{|c|} \hline \text{trivalent vertex} \\ \hline \end{array} \right) = q^{\frac{\gamma^2 - \alpha^2 - \beta^2 + (r-1)^2}{4}} N \left( \begin{array}{|c|} \hline \text{trivalent vertex} \\ \hline \end{array} \right),$$

$$(N \text{ h}) \quad N \left( \begin{array}{|c|} \hline \text{loop} \\ \hline \end{array} \right) = (-1)^{r-1} r q^{\alpha\beta},$$

$$(N \text{ i}) \quad N \left( \begin{array}{|c|} \hline \text{trivalent vertex} \\ \hline \end{array} \right) = \sum_{\gamma \in \alpha + \beta + H_r} d(\gamma) N \left( \begin{array}{|c|} \hline \text{trivalent vertex} \\ \hline \end{array} \right),$$

$$(N \text{ j}) \quad N \left( \begin{array}{|c|} \hline \text{6j-symbol} \\ \hline \end{array} \right) = \sum_{j_5 \in j_1 + j_6 + H_r} d(j_5)^{-1} \left| \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right| N \left( \begin{array}{|c|} \hline \text{6j-symbol} \\ \hline \end{array} \right).$$

Here the  $6j$ -symbol  $\left| \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right| = N \left( \begin{array}{|c|} \hline \text{6j-symbol} \\ \hline \end{array} \right)$  is given by:

$$\begin{aligned} \left| \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right| &= (-1)^{r-1+B_{165}} \frac{\{B_{345}\}! \{B_{123}\}!}{\{B_{246}\}! \{B_{165}\}!} \left[ \begin{array}{c} j_3 + r - 1 \\ A_{123} + 1 - r \end{array} \right] \left[ \begin{array}{c} j_3 + r - 1 \\ B_{354} \end{array} \right]^{-1} \times \\ &\times \sum_{z=m}^M (-1)^z \left[ \begin{array}{c} A_{165} + 1 \\ j_5 + z + r \end{array} \right] \left[ \begin{array}{c} B_{156} + z \\ B_{156} \end{array} \right] \left[ \begin{array}{c} B_{264} + B_{345} - z \\ B_{264} \end{array} \right] \left[ \begin{array}{c} B_{453} + z \\ B_{462} \end{array} \right] \end{aligned}$$

where  $A_{xyz} = \frac{j_x + j_y + j_z + 3(r-1)}{2}$ ,  $B_{xyz} = \frac{j_x + j_y - j_z + r - 1}{2}$ ,  $m = \max(0, \frac{j_3 + j_6 - j_2 - j_5}{2})$  and  $M = \min(B_{435}, B_{165})$ .

**Remark 1.1.** It can be shown that the above axioms determine the value of  $N(T)$  for any  $T \in \mathcal{G}$  (cf [12, Proposition 4.6] for a similar proof). In particular, the axioms can be used to reduce  $N(T)$  to a linear combination of  $6j$ -symbols which are in turn determined by the above formula. In Subsection 5.2 we will prove that these axioms are consistent.

**Remark 1.2.** The  $\mathbb{C} \setminus X_r$ -coloring  $c$  of  $T \in \mathcal{G}$  can be seen as a complex 1-chain. Its boundary  $\delta c$  is a 0-chain, i.e. a map from the set  $\mathcal{V}$  of trivalent vertices of  $T$  to  $\mathbb{C}$ . If  $\delta c$  has a value in  $\mathbb{C} \setminus H_r$ , then  $N(T) = 0$  because of Axioms (N a) and (N b). For a trivalent graph  $T$  and map  $\delta : \mathcal{V} \rightarrow H_r$  let  $C(T, \delta)$  be the set of colorings of  $T$  with boundary  $\delta$ . Then  $C(T, \delta)$  is a Zariski open subset

of an affine subspace of  $\mathbb{C}^{\{\text{edges}(T)\}}$ . As a function on  $C(T, \delta)$ ,  $\mathbf{N}$  is holomorphic. Finally, if the number of edges of  $T$  is greater or equal 2 then  $\mathbf{N}$  extends continuously to the closure of  $C(T, \delta)$ .

**Remark 1.3.** The formulas for the  $6j$ -symbols above have been computed in [6]. In particular, one can recover these symbols from Formula 1.16 of [6] where a color  $a \in \mathbb{C}$  in [6] corresponds to the color  $2a + 1 - r$  above. In the case when  $r$  is odd, these  $6j$ -symbols are computed in [13] but have a different normalization. There are several choices which give different normalizations of this invariant. One of them concerns the choice of  $\mathbf{d}$ : For any complex number  $c$ , one can replace  $\mathbf{d}$  with  $\tilde{\mathbf{d}} = c\mathbf{d}$  and define  $\tilde{\mathbf{N}}$  on a non empty graph  $T$  with  $v$  vertices by  $\tilde{\mathbf{N}}(T) = c^{1-v/2}\mathbf{N}(T)$ . Then  $\tilde{\mathbf{N}}$  satisfy an equivalent set of axioms.

We consider the following computation which will be useful later. Let  $\epsilon = 1$  if  $r$  is even and 0 otherwise and let  $\sigma$  be a formal linear combination of colors,  $\sigma = \sum_{k \in H_r} \mathbf{d}(\alpha + k)[\alpha + k]$ . For a graph with an edge colored by  $\sigma$ , we formally expand such a color as

$$\begin{aligned}
\mathbf{N} \left( \begin{array}{c} | \\ \sigma \curvearrowright \\ \downarrow \alpha + \epsilon \end{array} \right) &= \sum_{k \in H_r} \mathbf{d}(\alpha + k) \mathbf{N} \left( \begin{array}{c} | \\ \alpha + k \curvearrowright \\ \downarrow \alpha + \epsilon \end{array} \right) \\
&= \sum_{k \in H_r} \mathbf{d}(\alpha + k) (-1)^{r-1} r \left( q^{(r-1)^2} q^{-\frac{1}{2}(\alpha+\epsilon)^2} q^{-\frac{1}{2}(\alpha+k)^2} \right) q^{(\alpha+\epsilon)(\alpha+k)} \mathbf{d}(\alpha + \epsilon)^{-1} \mathbf{N} \left( \begin{array}{c} | \\ \alpha + \epsilon \end{array} \right) \\
&= \sum_{k \in H_r} (-1)^{r-1} r \frac{\{\alpha + k\}}{\{\alpha + \epsilon\}} q^{(r-1)^2} q^{\epsilon k - \frac{1}{2}k^2 - \frac{1}{2}\epsilon^2} \mathbf{N} \left( \begin{array}{c} | \\ \alpha + \epsilon \end{array} \right) \\
&= \frac{(-1)^{r-1} r q^{(r-1)^2 - \frac{1}{2}\epsilon^2}}{\{\alpha + \epsilon\}} \left( q^\alpha \sum_{k \in H_r} q^{\epsilon k - \frac{1}{2}k^2 + k} - q^{-\alpha} \sum_{k \in H_r} q^{\epsilon k - \frac{1}{2}k^2 - k} \right) \mathbf{N} \left( \begin{array}{c} | \\ \alpha + \epsilon \end{array} \right) \\
&= \frac{(-1)^{r-1} r q^{(r-1)^2 - \frac{1}{2}\epsilon^2}}{\{\alpha + \epsilon\}} (q^\alpha S_+ - q^{-\alpha} S_-) \mathbf{N} \left( \begin{array}{c} | \\ \alpha + \epsilon \end{array} \right).
\end{aligned}$$

where  $S_\pm = \sum_{k \in H_r} q^{\epsilon k - \frac{1}{2}k^2 \pm k}$ . Setting  $l = \frac{k+(r-1)}{2}$  one has

$$S_\pm = \sum_{l=0}^{r-1} q^{\epsilon(2l-(r-1)) - 2l^2 - \frac{1}{2}(r-1)^2 + 2(r-1)l \pm (2l-(r-1))} = q^{-\frac{1}{2}(r-1)^2 \mp (r-1) - \epsilon(r-1)} \sum_{l=0}^{r-1} q^{-2l^2 + (2\epsilon - 2 \pm 2)l}$$

which is a generalized quadratic Gauss sum and is computed for example in [4, Chapter 1]. Since  $-l^2 + \epsilon l = -(l-1)^2 - 2(l-1) - 1 + \epsilon(l-1) + \epsilon$  and  $q = \exp(\frac{i\pi}{r})$  then

$$S_+ = q^{-\frac{1}{2}(r-1)^2 - (r-1) - \epsilon(r-1)} \sum_{l=0}^{r-1} q^{-2l^2 + 2\epsilon l} = q^{-\frac{1}{2}(r-1)^2 + (r+1) - \epsilon(r-1)} \sum_{l=0}^{r-1} q^{-2(l-1)^2 - 4(l-1) - 2 + 2\epsilon(l-1) + 2\epsilon}$$

The last quantity equals  $q^{2\epsilon} S_-$ , thus we have:

$$\mathbf{N} \left( \begin{array}{c} | \\ \sigma \curvearrowright \\ \downarrow \alpha + \epsilon \end{array} \right) = (-1)^{r-1} r q^{(r-1)^2 - \frac{1}{2}\epsilon^2} \left( \frac{q^{\alpha+2\epsilon} - q^{-\alpha}}{\{\alpha + \epsilon\}} S_- \right) \mathbf{N} \left( \begin{array}{c} | \\ \alpha + \epsilon \end{array} \right) = \Delta_- \mathbf{N} \left( \begin{array}{c} | \\ \alpha + \epsilon \end{array} \right)$$

$$\text{where } \Delta_- = \begin{cases} 0 & \text{if } r \equiv 0 \pmod{4} \\ i(rq)^{\frac{3}{2}} & \text{if } r \equiv 1 \pmod{4} \\ (i-1)(rq)^{\frac{3}{2}} & \text{if } r \equiv 2 \pmod{4} \\ -(rq)^{\frac{3}{2}} & \text{if } r \equiv 3 \pmod{4} \end{cases} \quad \text{Similarly,}$$

$$\mathbf{N} \left( \begin{array}{c} \text{diagram of a crossing with a loop} \\ \downarrow \alpha + \epsilon \end{array} \right) = \Delta_+ \mathbf{N} \left( \begin{array}{c} \text{diagram of a crossing} \\ \downarrow \alpha + \epsilon \end{array} \right)$$

where  $\Delta_+$  is the complex conjugate of  $\Delta_-$ .

**1.3. Cohomology classes.** Here we give a characterization of a cohomology group which will be used throughout this paper. Let  $(G, +)$  be an abelian group. Let  $M$  be a compact connected oriented 3-manifold and  $T$  a framed graph in  $M$ . Let  $L$  be an oriented framed link in  $S^3$  which represents a surgery presentation of  $M$ . Consider the cohomology group  $H^1(M, G) \simeq \text{Hom}(H_1(M, \mathbb{Z}), G)$ . The meridians  $\{m_i\}_{i=1 \dots n_L}$  of the components of  $L$  generate  $H_1(M, \mathbb{Z})$  and their relations are given in these generators by the columns of the symmetric linking matrix  $\text{lk} = (\text{lk}_{ij})$ . Consequently,  $H^1(M, G) = \{(\phi_i) \in G^n : \text{lk} \cdot \phi = 0\}$ .

As we will now explain the cohomology group of  $M \setminus T$  can be described in terms of the homology of  $L \cup T$ . Let  $e_1, \dots, e_{n_L}$  be the oriented edges of  $L$ ,  $e_{n_L+1}, \dots, e_{n_L+n_T}$  be the oriented edges of  $T$  and  $m_i$  be the oriented meridian of  $e_i$ . We have  $H^1(M \setminus T, G) \simeq \text{Hom}(H_1(M \setminus T, \mathbb{Z}), G)$  where  $H_1(M \setminus T, \mathbb{Z})$  is generated by the meridians of all edges of  $L \cup T$ . Given a regular planar projection of  $L \cup T$ , we can define a linking matrix for the  $n_L + n_T$  edges of  $L \cup T$  (which is not an isotopy invariant) as follows: for  $i, j \in \{1, \dots, n_L + n_T\}$  let  $\text{lk}_{ij}$  be the algebraic number of crossings between the edges  $e_i$  and  $e_j$  with the edge  $e_j$  above the edge  $e_i$ . Graphically, this can be represented by

$$\text{lk}_{ij} \left( \begin{array}{c} \text{diagram of crossing } e_i \text{ over } e_j \\ \downarrow e_j \end{array} \right) = 1 \quad \text{lk}_{ij} \left( \begin{array}{c} \text{diagram of crossing } e_j \text{ over } e_i \\ \downarrow e_j \end{array} \right) = -1 \quad \text{lk}_{ij} \left( \begin{array}{c} \text{diagram of crossing } e_i \text{ over } e_j \\ \downarrow e_i \end{array} \right) = \text{lk}_{ij} \left( \begin{array}{c} \text{diagram of crossing } e_j \text{ over } e_i \\ \downarrow e_i \end{array} \right) = 0$$

Thus we may present:

$$H_1(M \setminus T, \mathbb{Z}) = \langle [m_i] \mid \sum_{j=1}^{n_L+n_T} \text{lk}_{ij} [m_j] = 0 \text{ and } r_v = 0 \text{ for all } i \in \{1, \dots, n_L\} \text{ and } v \rangle$$

where  $v$  ranges over all the vertices of  $T$  and  $r_v$  is the sum of the meridians of the edges incoming to  $v$  minus the sum of the meridians of the edges outgoing from  $v$ .

Let

$$(4) \quad \Phi : H^1(M \setminus T, G) \rightarrow H_1(L \cup T, G) = H_1(L, G) \oplus H_1(T, G)$$

be the map sending a cohomology class  $\omega$  to the chain  $\sum_i \omega(m_i) e_i$ ; clearly  $\Phi$  is injective. A cycle  $x = \sum_i x_i e_i$  representing a class in  $H_1(L \cup T, G)$  is in  $\text{Im}(\Phi)$  if and only if  $\sum_{j=1}^{n_L+n_T} \text{lk}_{ij} x_j = 0 \in G$ , for all  $i \in \{1, \dots, n_L\}$ .

For  $\omega \in H^1(M \setminus T, G)$  and  $L \cup T$  as above, the map  $g_\omega$  defined on the set of edges of  $L \cup T$  with values in  $G$  defined by  $g_\omega(e_i) = \omega(m_i)$  is called the  $G$ -coloring of  $L \cup T$  induced by  $\omega$ .

**1.4. The 3-manifolds invariants  $\mathbf{N}$  and  $\mathbf{N}^0$ .** In the rest of this section we use the notation of the last subsection with the additive group  $G = \mathbb{C}/2\mathbb{Z}$ . We also assume that the integer  $r$  is not congruent to 0 mod 4. Let  $\bar{X} = \mathbb{Z}/2\mathbb{Z} \subset \mathbb{C}/2\mathbb{Z}$ . Let  $M$  be a compact, connected oriented 3-manifold and  $T$  a framed trivalent graph in  $M$  whose edges are colored by elements of  $\mathbb{C} \setminus X_r$ . Let  $\omega \in H^1(M \setminus T, \mathbb{C}/2\mathbb{Z})$ . If  $M$  is presented as an integral surgery over a link  $L$  in  $S^3$  then  $\omega$

induces a  $\mathbb{C}/2\mathbb{Z}$ -coloring  $g_\omega$  on  $L \cup T$ . We say that  $(M, T, \omega)$  is a *compatible triple* if for each edge  $e$  of  $T$  its  $\mathbb{C} \setminus X_r$ -color  $c(e)$  satisfies  $c(e) + r - 1 \equiv g_\omega(e) \pmod{2\mathbb{Z}}$ . Note that this definition does not depend on the surgery presentation of  $M$ . A surgery presentation via  $L \subset S^3$  for a compatible tuple  $(M, T, \omega)$  is *computable* if one of the following two conditions holds: (1)  $g_\omega(e) \in \mathbb{C}/2\mathbb{Z} \setminus \overline{X}$  for all edges  $e$  of  $L$ , or (2)  $L = \emptyset$  and  $T \neq \emptyset$ .

For  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$  we define the Kirby color  $\Omega_\alpha \in \text{Span}_{\mathbb{C}} \langle [x] \mid x \in \mathbb{C} \rangle$  by

$$\Omega_\alpha = \sum_{k \in H_r} d(\alpha + k)[\alpha + k].$$

If  $\bar{\alpha}$  is the image of  $\alpha + r - 1$  in  $\mathbb{C}/2\mathbb{Z}$  we say that  $\Omega_\alpha$  has degree  $\bar{\alpha}$ . We can “color” a knot  $K$  with a Kirby color  $\Omega_\alpha$ : let  $K(\Omega_\alpha)$  be the formal linear combination of knots  $\sum_{k \in H_r} d(\alpha + k)K(\alpha + k)$  where  $K(\alpha + k)$  is the knot  $K$  colored with  $\alpha + k$ . If  $\bar{\alpha} \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$ , by  $\Omega_{\bar{\alpha}}$ , we mean any Kirby color of degree  $\bar{\alpha}$ .

We will prove all of the theorems and propositions of this section in Section 5.

**Theorem 1.4.** *If  $L$  is a link which gives rise to a computable surgery presentation of a compatible triple  $(M, T, \omega)$  then*

$$\mathbf{N}(M, T, \omega) = \frac{\mathbf{N}(L \cup T)}{\Delta_+^p \Delta_-^s}$$

*is a well defined topological invariant (i.e. depends only of the homeomorphism class of the triple  $(M, T, \omega)$ ), where  $(p, s)$  is the signature of the linking matrix of the surgery link  $L$  and for each  $i$  the component  $L_i$  is colored by a Kirby color of degree  $g_\omega(L_i)$ .*

Next we show that computable surgery presentation exist in several situations. To do this we need the following definition. The inclusion  $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{C}/2\mathbb{Z}$  induces an injective map

$$H^1(M \setminus T, \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(H_1(M \setminus T, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \text{Hom}(H_1(M \setminus T, \mathbb{Z}), \mathbb{C}/2\mathbb{Z}) \cong H^1(M \setminus T, \mathbb{C}/2\mathbb{Z}).$$

We say that a cohomology class  $\omega \in H^1(M \setminus T, \mathbb{C}/2\mathbb{Z})$  is *integral* if it is in the image of this map.

**Proposition 1.5.** *Let  $(M, T, \omega)$  be a compatible triple where the cohomology class  $\omega$  is not integral. Then there exists a surgery presentation of  $(M, T, \omega)$  which is computable. In particular, the triple  $(M, T, \omega)$  has a computable surgery presentation if  $T \neq \emptyset$  and  $T$  has an edge whose color is in  $\mathbb{C} \setminus \mathbb{Z}$ .*

The proposition (whose proof is constructive) implies that when  $\omega$  is not integral one may construct a computable presentation of  $(M, T, \omega)$  and hence apply Theorem 1.4. The proposition does not apply to the following case: when  $\omega$  is integral,  $T$  is non-empty and all the edges of  $T$  are colored by a multiple of  $r$ . The following construction allows one to extend the definition of  $\mathbf{N}$  to this case and provides a method for computing  $\mathbf{N}$  when computable presentations exist. With this in mind, we consider the situation when  $T \neq \emptyset$ .

**Definition 1.6** (*H-Stabilization*). Recall the notation  $X_r = \mathbb{Z} \setminus r\mathbb{Z} \subset \mathbb{C}$ . Let  $H(\alpha, \beta)$  be a long Hopf link in  $\mathbb{R}^3$  whose circle component is colored by  $\alpha \in \mathbb{C} \setminus X_r$  and whose long component is colored by  $\beta \in \mathbb{C} \setminus X_r$ . Let  $(M, T, \omega)$  be a compatible triple where  $T \neq \emptyset$ ,  $e$  is an edge of  $T$  colored by  $\beta$  and  $m$  is the meridian of  $e$ . A *H-stabilization* of  $(M, T, \omega)$  along  $e$  is a compatible triple  $(M, T_H, \omega_H)$  such that

- $T_H = T \cup m$  and  $m$  is colored by  $\alpha \in \mathbb{C} \setminus X_r$ ,
- $\omega_H$  is the unique element of  $H^1(M \setminus (T \cup m); \mathbb{C}/2\mathbb{Z})$  such that  $\omega_H(m) = \bar{\alpha}$  is equal to the image of  $\alpha + r - 1$  in  $\mathbb{C}/2\mathbb{Z}$  and  $(\omega_H)|_{M \setminus (T \cup D)} = \omega$  where  $D$  is a disc bounded by  $m$  and intersecting once  $e$ .



**Theorem 1.7.** *If  $(M, T, \omega)$  is a compatible triple and  $T \neq \emptyset$  then there exists a  $H$ -stabilization of  $(M, T, \omega)$  admitting a computable surgery presentation. Let  $(M, T_H, \omega_H)$  be such a  $H$ -stabilization and let  $L$  be a link which gives rise to a computable surgery presentation of  $(M, T_H, \omega_H)$  then*

$$\mathbf{N}(M, T, \omega) = \frac{\mathbf{d}(\beta)\mathbf{N}(L \cup T_H)}{(-1)^{r-1}rq^{\alpha\beta}\Delta_+^p\Delta_-^s}$$

*is a well defined topological invariant (i.e. depends only of the homeomorphism class of the triple  $(M, T, \omega)$ ), where a component  $L_i$  of  $L$  is colored by a Kirby color of degree  $g_\omega(L_i)$  and  $(p, s)$  is the signature of the linking matrix of the surgery link  $L$ . Moreover, if an edge of  $T$  is colored by an element of  $\mathbb{C} \setminus \mathbb{Z}$  then the above quantity coincides with the invariant of Theorem 1.4.*

Theorems 1.4 and 1.7 do not apply when  $\omega$  is integral and  $T$  is empty. To cover this case we define a second invariant  $\mathbf{N}^0$ . This invariant naturally completes the definition of  $\mathbf{N}$  in the following sense:  $\mathbf{N}^0(M, T, \omega)$  is defined for any triple  $(M, T, \omega)$  but it is zero if a computable surgery presentation of  $(M, T, \omega)$  exists. To do so, we define the connected sum of two triples.

Let  $(M_1, T_1, \omega_1)$  and  $(M_2, T_2, \omega_2)$  be compatible triples. Let  $M_3 = M_1 \# M_2$  is the connected sum along balls not intersecting  $T_1$  and  $T_2$ . Let  $T_3 = T_1 \sqcup T_2$ . For  $i = 1, 2$ , let  $B_i^3$  be a 3-ball in  $M_i \setminus T_i$ . Then we have

$$H_1(M_3 \setminus T_3; \mathbb{Z}) \cong H_1(M_1 \setminus (B_1^3 \sqcup T_1); \mathbb{Z}) \oplus H_1(M_2 \setminus (B_2^3 \sqcup T_2); \mathbb{Z}) \cong H_1(M_1 \setminus T_1; \mathbb{Z}) \oplus H_1(M_2 \setminus T_2; \mathbb{Z})$$

where the first isomorphism is induced by a Mayer-Vietoris sequence and the second isomorphism comes from excision. These maps induce an isomorphism

$$H^1(M_3 \setminus T_3, \mathbb{C}/2\mathbb{Z}) \rightarrow H^1(M_1 \setminus T_1, \mathbb{C}/2\mathbb{Z}) \oplus H^1(M_2 \setminus T_2, \mathbb{C}/2\mathbb{Z}).$$

Let  $\omega_3$  be the unique element of  $H^1(M_3 \setminus T_3, \mathbb{C}/2\mathbb{Z})$  such that  $\omega_3$  restricts through the above isomorphism to both  $\omega_1$  and  $\omega_2$ . Define the connected sum of  $(M_1, T_1, \omega_1)$  and  $(M_2, T_2, \omega_2)$  as

$$(5) \quad (M_1, T_1, \omega_1) \# (M_2, T_2, \omega_2) = (M_3, T_3, \omega_3).$$

For  $\alpha \in \mathbb{C} \setminus X_r$ , let  $u_\alpha$  be the unknot in  $S^3$  colored by  $\alpha$ . Let  $\omega_\alpha$  be the unique element of  $H^1(S^3 \setminus u_\alpha; \mathbb{C}/2\mathbb{Z})$  such that  $(S^3, u_\alpha, \omega_\alpha)$  is a compatible triple. Remark that  $\mathbf{N}(S^3, u_\alpha, \omega_\alpha) = \mathbf{d}(\alpha)$ .

**Theorem 1.8.** *Let  $(M, T, \omega)$  be a compatible triple. Define*

$$\mathbf{N}^0(M, T, \omega) = \frac{\mathbf{N}((M, T, \omega) \# (S^3, u_\alpha, \omega_\alpha))}{\mathbf{d}(\alpha)}.$$

*Then  $\mathbf{N}^0(M, T, \omega)$  is a well defined topological invariant (i.e. depends only of the homeomorphism class of the compatible triple  $(M, T, \omega)$ ). Moreover, if  $(M, T, \omega)$  or an  $H$ -stabilization of  $(M, T, \omega)$  has a computable surgery presentation then  $\mathbf{N}^0(M, T, \omega) = 0$ .*

Notice that the above theorems can be used to define a possibly non-zero invariant for every compatible triple  $(M, T, \omega)$ . In particular, if  $T \neq \emptyset$  or if  $\omega$  is not integral then the invariant  $\mathbf{N}$  is well defined (in this case  $\mathbf{N}^0$  is defined but zero), otherwise,  $\mathbf{N}^0$  is well defined and possibly non-zero (for example it is non-zero on the Poincaré sphere, see Subsection 2.4). In Section 5 we will prove all of the theorems of this section. In Section 2 we give several examples that show the invariants  $\mathbf{N}$  and  $\mathbf{N}^0$  are computable and non-trivial. For other properties of these invariants see Section 3.4.

## 2. EXAMPLES AND APPLICATIONS

**2.1. Distinguishing lens spaces.** In Remark 3.9 of [20] it was observed that the standard Reshetikhin-Turaev 3-manifold invariants arising from quantum  $\mathfrak{sl}(2)$  cannot distinguish the lens spaces  $L(65, 8)$  and  $L(65, 18)$ . In this subsection, we show that for  $r = 3$  and  $T = \emptyset$ , the invariant  $N$  can distinguish these two manifolds. Thus, the invariants defined in this paper are independent of the standard Reshetikhin-Turaev 3-manifold invariants. Interestingly, this result depends on the cohomological data of the invariant. Since it can be difficult to compare two elements of the isomorphic spaces  $H_1(L(65, 8); \mathbb{C}/2\mathbb{Z})$  and  $H_1(L(65, 18); \mathbb{C}/2\mathbb{Z})$  we consider the sum over the whole cohomological space (except the zero element):

$$S_r(L(p, q)) = \sum_{\omega \in H^1(L(p, q); \mathbb{C}/2\mathbb{Z}), \omega \neq 0} N(L(p, q), \emptyset, \omega)$$

where this sum is finite since  $H_1(L(p, q); \mathbb{C}/2\mathbb{Z})$  is isomorphic to  $\mathbb{Z}/p\mathbb{Z}$ .

Let  $C(a_1, \dots, a_n)$  be the chain link whose components are oriented unknots  $L_1, \dots, L_n$  with framings  $a_1, \dots, a_n$  (i.e.  $L_i$  and  $L_j$  form a Hopf link if  $j = i \pm 1$  and a split link otherwise) and let  $m_i$  be oriented meridian of  $L_i$ . It is a standard fact that surgery on  $C(a_1, \dots, a_n)$  provides  $L(p, q)$  when  $\frac{p}{q} = a_1 - \frac{1}{a_2 - \dots - \frac{1}{a_n}}$ . Let  $\text{lk}$  be the linking matrix of  $C(a_1, \dots, a_n)$  with respect to the components  $L_i$ . Color  $L_i$  with  $\alpha_i \in \mathbb{C} \setminus 2\mathbb{Z}$  such that  $\sum_j \text{lk}_{ij} \bar{\alpha}_j = 0$ , for all  $i$ , where  $\bar{\alpha}_i$  is the image of  $\alpha_i + r - 1$  in  $\mathbb{C}/2\mathbb{Z}$  and this sum is considered as an equation in  $\mathbb{C}/2\mathbb{Z}$ . The coloring  $\{\alpha_i\}$  represents the cohomology class  $\omega \in H^1(L(p, q); \mathbb{C}/2\mathbb{Z})$  determined by  $\omega([m_i]) = \bar{\alpha}_i$ . Using the axioms of Section 1.2, one can show that  $N(L(p, q), \emptyset, \omega)$  is equal to:

$$(6) \quad \sum_{k_1, \dots, k_n \in H_r} \frac{d(\alpha_1 + k_1) d(\alpha_n + k_n) \prod_{j=1}^n q^{a_i \left( \frac{(\alpha_i + k_i)^2 - (r-1)^2}{2} \right)} \prod_{j=1}^{n-1} (-1)^{r-1} r q^{(\alpha_j + k_j)(\alpha_{j+1} + k_{j+1})}}{\Delta_+^{n_+} \Delta_-^{n_-}}$$

where  $n_+$  (resp.  $n_-$ ) is the number of positive (resp. negative) eigenvalues of  $\text{lk}$  and  $q = \exp(\frac{\sqrt{-1}\pi}{r})$ .

Since  $H_1(L(p, q); \mathbb{C}/2\mathbb{Z})$  is isomorphic to  $\mathbb{Z}/p\mathbb{Z}$  and  $[m_1]$  is a generator of this homology group then for any  $\omega \in H^1(L(p, q); \mathbb{C}/2\mathbb{Z})$  the value  $\omega([m_1]) \in \mathbb{C}/2\mathbb{Z}$  is an integer multiple of  $\frac{2}{p}$ . Conversely, if  $\bar{\alpha}_1 \in \mathbb{C}/2\mathbb{Z}$  is any integer multiple of  $\frac{2}{p}$  we can find a unique cohomology class  $\omega \in H^1(L(p, q); \mathbb{C}/2\mathbb{Z})$  such that  $\omega([m_1]) = \bar{\alpha}_1$ . Thus, the elements of  $H^1(L(p, q); \mathbb{C}/2\mathbb{Z})$  are in one to one correspondence with integer multiples of  $\frac{2}{p}$  in  $\mathbb{C}/2\mathbb{Z}$ . Moreover, for  $\omega \in H^1(L(p, q); \mathbb{C}/2\mathbb{Z})$  one can compute the values  $\bar{\alpha}_i = \omega([m_i])$  by solving the linear equations  $\sum \text{lk}_{i,j} \bar{\alpha}_j = 0$  (in  $\mathbb{C}/2\mathbb{Z}$ ) in terms of  $\bar{\alpha}_1$ . Combining this fact with Equation (6) it is easy to use computer algebra software to numerically compute  $S_r(L(p, q))$ . In particular, one can show that  $S_3(L(65, 8)) \neq S_3(L(65, 18))$  (for instance use the surgery presentations over  $C(8, -8)$  and  $C(4, 3, 2, -3)$ , respectively). This was checked independently by the first and third authors.

**2.2. On the volume conjecture.** In [7] the first author proved a version of the volume conjecture for an infinite class of hyperbolic links called “fundamental hyperbolic links”. In this section we show that a similar conjecture holds for the invariants introduced in this paper. The main advantage of this new approach is that now one may formulate a version of the volume conjecture for any link or knot in a 3-manifold using our invariants. This was not possible using the approach of [7] where the fact that the ambient manifold was a connected sum of  $S^1 \times S^2$  or a sphere.

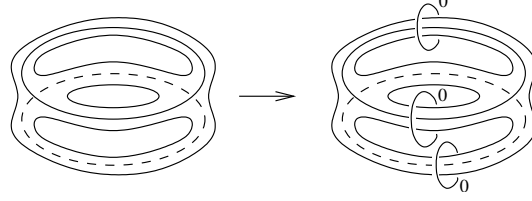



FIGURE 1. An instance of construction of fundamental hyperbolic link.

**Definition 2.1** (Fundamental hyperbolic links). A *fundamental hyperbolic link*  $L$  is a link contained in a 3-manifold  $M$  diffeomorphic to a connected sum of  $k \geq 2$  copies of  $S^2 \times S^1$  and obtained by the following procedure:

- (1) let  $\Gamma$  be a graph in  $S^3$  with  $k - 1$  four-valent vertices and let  $T \subset \Gamma$  be a maximal tree;
- (2) in a diagram of  $\Gamma$  replace each vertex of  $\Gamma$  by the diagram ;
- (3) connect the new three-uples of boundary points (with any permutation) following the edges of  $\Gamma$  and let  $L$  be the resulting link in  $S^3$ ;
- (4) put 0-framed meridians around the each of the three-uple of strands passing along the edges of  $\Gamma \setminus T$ ;
- (5) perform surgery on the 0-framed meridians to realize  $L$  as a link in the connected sum of  $k$  copies of  $S^2 \times S^1$ .

A fundamental hyperbolic link has no natural orientation but such a link may be equipped with a natural framing (see [8] for more details). However, for our purposes we can choose arbitrarily an orientation and a framing. Indeed, we will consider the norm of the invariants of triples  $(M, L, \omega_r)$  where the induced coloring on  $L$  is 0 and so by Axiom (N a) the choice of an orientation is irrelevant and by Axiom (N f) the framing changes the value of the invariant by a multiplicative factor of the form  $q^{\pm \frac{(r-1)^2}{2}}$  and so does not change the norm of the invariant.

**Example 2.2.** Let  $\Gamma$  be the graph with 2 vertices and 4 edges connecting them. Apply the procedure of Definition 2.1 to  $\Gamma$ . In this procedure one can connect the three-uples of boundary points in such a way that one obtains the link in left side of Figure 1. In the right side of the figure the 0-framed meridians have been added. After surgery the resulting link is contained in the connected sum of 3 copies of  $S^2 \times S^1$ .

The following theorem was proved in [8].

**Theorem 2.3** ([8]). *Let  $L$  be a fundamental hyperbolic link in  $M = \#_k S^2 \times S^1$ . Then  $M \setminus L$  admits a complete hyperbolic structure whose volume is*

$$2(k-1)\text{VolOct} = 16(k-1)\Lambda\left(\frac{\pi}{4}\right)$$

where  $\text{VolOct}$  is the volume of a regular ideal octahedron and  $\Lambda(x) = \int_0^x -\log|2\sin(t)|dt$  is the Lobachevsky function. Moreover, for any link  $T$  in an oriented 3-manifold  $N$  there exists a fundamental hyperbolic link  $L = L' \cup L'' \subset \#_k S^2 \times S^1$  (for some  $k$ ) such that the result of an integral surgery on  $L'$  is the manifold  $N$  and the image of  $L''$  is  $T$ .

The following theorem is the main result of this subsection.

**Theorem 2.4** (A version of the Volume Conjecture). *Let  $L$  be a fundamental hyperbolic link in  $M = \#_k S^2 \times S^1$  colored by  $0 \in \mathbb{C} \setminus X_r$ . For each odd integer  $r$ , let  $\omega_r \in H^1(M \setminus L; \mathbb{C}/2\mathbb{Z})$  a*

cohomology class such that it associated  $\mathbb{C}/2\mathbb{Z}$ -coloring satisfies  $g_{\omega_r}(m_i) = 0$ , for all  $i$  (where  $m_i$  is the oriented meridian of the  $i^{\text{th}}$ -component of  $L$ ). Then

$$\lim_{\substack{r \rightarrow \infty \\ r \text{ odd}}} \frac{2\pi}{r} \log(|\mathbf{N}(M, L, \omega_r)|) = \text{Vol}(M \setminus L).$$

Before we prove Theorem 2.4 we state the following proposition whose proof is given in Subsection 5.4.

**Proposition 2.5.** *Let  $r$  be an odd integer and  $a \in \mathbb{C} \setminus \mathbb{Z}$ . Let  $\alpha, \beta, \gamma \in \mathbb{C} \setminus X_r$  such that  $\alpha + \beta + \gamma = 0$ . Then*

$$(7) \quad F \left( \begin{array}{c} \text{Diagram with 3 strands, a loop labeled } \Omega_a, \text{ and arrows labeled } \alpha, \beta, \gamma \end{array} \right) = r^3 F \left( \begin{array}{c} \text{Diagram with 3 strands and arrows labeled } \alpha, \beta, \gamma \end{array} \right).$$

*Proof of Theorem 2.4.* Start with a presentation of  $(M, L)$  as in Definition 2.1 and let  $T, \Gamma$  be as in this definition. We assume first that this presentation is computable. First, one has  $\mathbf{N}(M, L, \omega_r) = r^{3k} \mathbf{N}(\tilde{\Gamma})$  where  $\tilde{\Gamma}$  is the trivalent graph obtained by applying Proposition 2.5 to each three-uple of strands in  $L$  encircled by a 0-framed meridian. In particular  $\tilde{\Gamma}$  contains twice as many vertices as there are 0-framed meridians and all edges of  $\tilde{\Gamma}$  are colored by 0. One can realize  $\tilde{\Gamma}$  in  $S^3$  so that it coincides with the diagram of  $L$  except near the 0-framed meridians. The maximal tree  $T$  has  $k - 1$  4-valent vertices and  $k - 2$  internal edges. Now, every internal edge of  $T$  corresponds to three strands of  $\tilde{\Gamma}$  connecting two disjoint subgraphs of  $\tilde{\Gamma}$  as in the left hand side of (N e). Applying Axiom (N e) to all the three-uple of strands of  $\tilde{\Gamma}$  corresponding to internal edges of  $T$  (i.e. those not encircled by 0-framed meridians) one gets that  $\mathbf{N}(\tilde{\Gamma})$  is equal to the product of the values of  $\mathbf{N}$  on  $k - 1$  tetraedral graphs (one for each vertex of  $T$ ). Axioms (N f) and (N g) imply that the modulus of  $\mathbf{N}$  of these tetraedral graphs is independent of the framing and of the cyclic ordering at its vertices, and thus is equal to the modulus of  $\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}_r$ . Hence we get

$$|\mathbf{N}(M, L, \omega_r)| = r^{3k} \left| \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}_r \right|^{k-1}.$$

To prove the final statement let's use Lemma 1.15 of [6] in the case when  $a = b = c = d = e = f = \frac{r-1}{2}$  (in our notation all the colors correspond to 0). Then we have:

$$\begin{aligned} \left| \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}_r \right| &= (-1)^{r-1} \sum_{z=0}^{\frac{r-1}{2}} \left[ \begin{array}{c} \frac{r-1}{2} \\ z \end{array} \right] \left[ \begin{array}{c} \frac{r-1}{2} + z \\ \frac{r-1}{2} \end{array} \right]^2 \left[ \begin{array}{c} r-1-z \\ \frac{r-1}{2} \end{array} \right] = \\ &= (-1)^{r-1} \sum_{z=0}^{\frac{r-1}{2}} \left[ \begin{array}{c} \frac{r-1}{2} \\ z \end{array} \right] \left[ \begin{array}{c} \frac{r-1}{2} \\ \frac{r-1}{2} - z \end{array} \right]^2 \left[ \begin{array}{c} \frac{r-1}{2} \\ z \end{array} \right] = r^2 \sum_{z=0}^{\frac{r-1}{2}} (\{z\}! \{ \frac{r-1}{2} - z \}!)^{-4} \end{aligned}$$

Where we used the equalities

$$\{a\}! \{r-1-a\}! = \sqrt{-1}^{r-1} r, \text{ and } \left[ \begin{array}{c} a \\ b \end{array} \right] = \left[ \begin{array}{c} r-1-b \\ r-1-a \end{array} \right]$$

which hold whenever  $a, b, a-b \in \{0, \dots, r-1\}$ . It is now a standard analysis to check that the summands are all positive real numbers, that the term growing faster is that corresponding to  $z = \lfloor \frac{r-1}{4} \rfloor$  and that its growth rate is  $\exp(\frac{8r}{\pi} \Lambda(\frac{\pi}{4}))$ . One concludes by Theorem 2.3. Finally, the

proof when the presentation is not computable, that is when one of the  $\mathbb{C}/2\mathbb{Z}$  color of the framed meridian is 0 or 1, can be deduced by continuity from Subsection 2.5. 2.5

**Question 2.6** (A version of the Volume Conjecture for links in manifolds). *Let  $L$  be a 0-colored link in a compact, oriented 3-manifold  $M$  such that  $M \setminus L$  has a complete hyperbolic metric with volume  $\text{Vol}(M \setminus L)$ . For each odd integer  $r$ , let  $\omega_r \in H^1(M \setminus L; \mathbb{C}/2\mathbb{Z})$  be the zero cohomology class. Does the equality*

$$\lim_{r \rightarrow \infty} \frac{2\pi}{r} \log |\mathbf{N}(M, L, \omega_r)| = \text{Vol}(M \setminus L)$$

*hold?*

The above question includes the usual volume conjecture which corresponds to the case  $M = S^3$ . Indeed,  $\mathbf{N}(S^3, L, \omega)$  is the ADO invariant of  $L$  (for a link in  $S^3$  or in a homology sphere, the  $\mathbb{C}$  coloring of the link uniquely determines a cohomology class  $\omega \in H^1(M \setminus L; \mathbb{C}/2\mathbb{Z})$  such that the triple is compatible). With our notation, the color 0 corresponds to the specialization of ADO invariant which H. Murakami and J. Murakami showed is the Kashaev invariant (see [28]).

**Remark 2.7.** Observe that Theorem 2.4 states something stronger than what is asked in the above question. Indeed the cohomology classes considered in the theorem are not necessarily zero: only the coloring they induce on  $L$  is zero. So one may ask a stronger question considering this larger class of cohomology classes.

**Remark 2.8.** The restriction to odd  $r$  is coherent both with the statement of Theorem 2.4 and with the results of [27] concerning the standard version of the volume conjecture.

At this point the following natural question is not supported by any evidence.

**Question 2.9** (A version of the Volume Conjecture for closed manifolds). *Let  $M$  be a closed oriented 3-manifold admitting a complete hyperbolic metric with volume  $\text{Vol}(M)$ . For each odd integer  $r$ , let  $\omega_r \in H^1(M; \mathbb{C}/2\mathbb{Z})$  be the zero cohomology class. Does the equality*

$$\lim_{r \rightarrow \infty} \frac{2\pi}{r} \log |\mathbf{N}^0(M, \emptyset, \omega_r)| = \text{Vol}(M)$$

*hold?*

**2.3. Surgeries on  $T(2, 2n+1)$  torus knots.** Let  $K_{2n+1}^f$  be the closure of the two strand braid  $\sigma_{1,2}^{2n+1}$  with  $f$  additional full twists with respect to the blackboard framing. In other words,  $K_{2n+1}^f$  is a torus knot of type  $T(2, 2n+1)$  with  $f$  additional twists. For example, when  $f = -2$  and  $n = 1$  then  $K_3^{-2}$  is the trefoil knot with framing  $+1$ . Let  $K_{2n+1}^f(\alpha)$  be  $K_{2n+1}^f$  colored by  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ .

Applying Relation (N i) to the two parallel strands of  $K_{2n+1}^f(\alpha)$  and then applying Relations (N g) and (N f) we have

$$(8) \quad \mathbf{N}(K_{2n+1}^f(\alpha)) = \sum_{k \in H_r} q^{ft_\alpha + \frac{2n+1}{2}(-2t_\alpha + t_{2\alpha+k})} \mathbf{d}(2\alpha + k)$$

where  $t_\alpha = \frac{\alpha^2 - (r-1)^2}{2}$ . In particular, for the trefoil  $K_3^{-2}$  and  $r = 5$  we have:

$$(9) \quad \mathbf{N}(K_3^{-2}(\alpha)) = \frac{5q^{\alpha^2/2} q^3 \{3\alpha\} + q\{5\alpha\} - q^{-1}\{9\alpha\}}{\{5\alpha\}}$$

Let  $M_{f,n}$  be the manifold obtained by doing surgery along  $K_{2n+1}^f$ . We will now compute  $\mathbf{N}(M_{f,n})$ . Suppose that  $f + 2n + 1 \neq 0$ ; since  $H_1(M_{f,n}; \mathbb{Z}) = \mathbb{Z}/(f + 2n + 1)\mathbb{Z}$  (and a generator is represented by the meridian of  $K_{2n+1}^f$ ), a cohomology class  $\omega \in H^1(M_{f,n}; \mathbb{C}/2\mathbb{Z})$  is determined by

its value  $\bar{\alpha} \in \mathbb{C}/2\mathbb{Z}$  on a generator of  $H_1(M_{f,n}; \mathbb{Z})$ ; clearly such a value multiplied by  $f + 2n + 1$  must be 0 in  $\mathbb{C}/2\mathbb{Z}$ . Then choosing  $\alpha \in \mathbb{C}$  such that  $\alpha + r - 1 \equiv \frac{2}{f+2n+1} \pmod{2\mathbb{Z}}$  one can compute  $N(M_{f,n}, \emptyset, \omega)$  where  $\bar{\alpha}$  is the class of  $\alpha + r - 1$  in  $\mathbb{C}/2\mathbb{Z}$ :

$$N(M_{f,n}, \emptyset, \omega) = \frac{\sum_{k \in H_r} d(\alpha + k) N(K_{2n+1}^f(\alpha + k))}{\Delta_{\text{sign}(f+2n+1)}}.$$

**2.4. Non-triviality of  $N^0$  on Poincaré sphere.** In this subsection we give several formulas for  $N^0(M, \emptyset, 0)$  when  $M$  is a 3-manifold obtain by a surgery on a knot. Then we use these formulas to show that  $N^0(P, \emptyset, 0) \neq 1$  where  $P$  is the Poincaré sphere.

Let  $M$  be a manifold obtained by surgery on a knot  $K$  with framing  $f \neq 0$ . Let  $K(\alpha)$  be the knot  $K$  colored with  $\alpha \in \mathbb{C}$ . We have

**Proposition 2.10.** *Define  $P(\alpha) = \sum_{k \in H_r} N(K(\alpha + k))$ . Then  $q^{-\frac{f}{2}\alpha^2} P(\alpha)$  is a Laurent polynomial in  $q^{\pm\alpha}$ . In particular, it is continuous at  $\alpha \in \mathbb{Z}$ . Furthermore,*

$$N^0(M, \emptyset, 0) = \frac{1}{\Delta_{\text{sign}(f)}} \sum_{k \in H_r} q^k P(k) = \frac{1}{\Delta_{\text{sign}(f)}} \sum_{k \in H_r} q^{-k} P(k).$$

*Proof.* Let  $DK(\alpha, \beta)$  be the 2-cable of  $K$  whose components are colored with  $\alpha$  and  $\beta$ . In order to compute  $N^0(M, \emptyset, 0)$ , we consider  $K$  colored by  $0 \in \mathbb{C}/2\mathbb{Z}$  unlinked with the unknot  $u_\alpha$  colored by  $\alpha$ . Sliding  $u_\alpha$  over  $K$  one obtains  $DK(\alpha, \Omega_{-\bar{\alpha}})$  (where  $\Omega_{-\bar{\alpha}}$  is a Kirby color of degree  $-\bar{\alpha}$ ). Observe now that (applying the fusion rule (N g) twice) one gets

$$N(DK(\alpha, \beta)) = \sum_{k \in H_r} N(K(\alpha + \beta + k)) = P(\alpha + \beta).$$

Moreover, one can prove that  $N(DK(\alpha, \beta))$  is a Laurent polynomial in  $q^\alpha, q^\beta, q^{\alpha^2}, q^{\beta^2}$  and thus we have for any  $h \in \mathbb{Z}$ ,

$$N(DK(\alpha, h - \alpha)) = \lim_{\beta \rightarrow h - \alpha} \sum_{k \in H_r} N(K(\alpha + \beta + k)) = P(h).$$

Now by definition of  $\Omega_{-\bar{\alpha}}$  and of  $N^0$ ,

$$N^0(M, \emptyset, 0) = \frac{1}{\Delta_{\text{sign}(f)} d(\alpha)} \sum_{h \in H_r} d(h - \alpha) N(DK(\alpha, h - \alpha)).$$

Thus,

$$\begin{aligned} \Delta_{\text{sign}(f)} N^0(M, \emptyset, 0) &= \frac{1}{\{\alpha\}} \sum_{h \in H_r} \{\alpha - h\} P(h) \\ &= \frac{q^\alpha}{q^\alpha - q^{-\alpha}} \sum_{h \in H_r} q^{-h} P(h) - \frac{q^{-\alpha}}{q^\alpha - q^{-\alpha}} \sum_{h \in H_r} q^h P(h). \end{aligned}$$

And the result follows from the fact that  $N^0(M, \emptyset, 0)$  does not depend on  $\alpha$ . 2.10

As in the last proposition it can be shown that there exists a Laurent polynomial  $\tilde{K}(X) \in \mathbb{C}[X, X^{-1}]$  such that

$$(10) \quad N(K(\alpha)) = \theta_\alpha d(\alpha) \tilde{K}(q^\alpha)$$

where  $\theta_\alpha = q^{\frac{1}{2}(\alpha^2 - (r-1)^2)}$  is a factor coming from the framing. We will use this fact to give another formula for  $N^0(M, \emptyset, 0)$ .

**Proposition 2.11.** *Assume that  $r$  is odd. Let  $\tilde{K}(X)$  be the Laurent polynomial discussed above. Suppose that  $N(K(\alpha)) = N(K(-\alpha))$  or equivalently that  $\tilde{K}(X^{-1}) = \tilde{K}(X)$ , then*

$$N^0(M, \emptyset, 0) = \frac{rf\theta_0^f}{2\{1\}\Delta_{\text{sign}(f)}} \sum_{n \in H_r} q^{2fn^2} \{2n\}^2 \tilde{K}(q^{2n}).$$

*Proof.* From Proposition 2.10, we have

$$\begin{aligned} S = \Delta_{\text{sign}(f)} N^0(M, \emptyset, 0) &= \sum_{\ell \in H_r} q^\ell P(\ell) = \lim_{\varepsilon \rightarrow 0} \sum_{k, \ell \in H_r} q^\ell N(K(\varepsilon + k + \ell)) \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{n=1-r}^{r-1} \sum_{\substack{h, \ell \in H_r \\ h + \ell = 2n}} q^\ell N(K(\varepsilon + 2n)) \end{aligned}$$

The sum of  $q^\ell$  for  $h, \ell \in H_r$  with  $h + \ell = 2n$  has as many terms as there are possible value of  $\ell$  :  $\max(1-r, 1-r+n) \leq \ell \leq \min(r-1, r-1+n)$ . This sum is equal to  $q^n \frac{\{r-|n|\}}{\{1\}} = q^n \frac{\{|n|\}}{\{1\}}$ , so

$$\begin{aligned} S &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\{1\}} \sum_{n=1-r}^{r-1} q^n \{|n|\} N(K(\varepsilon + 2n)) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\{1\}} \sum_{n=1-r}^{r-1} q^n \{|n|\} \frac{1}{2} (N(K(2n + \varepsilon)) + N(K(2n - \varepsilon))) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\{1\}} \sum_{n=1-r}^{r-1} q^n \{|n|\} (N(K(\varepsilon + 2n)) + N(K(\varepsilon - 2n))) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\{1\}} \sum_{n=1}^{r-1} \{2n\} (N(K(\varepsilon + 2n)) + N(K(\varepsilon - 2n))) \end{aligned}$$

Here the second equality is the average of the right and left limits, the third equality comes from the fact that  $N(K(\alpha)) = N(K(-\alpha))$  and the last equality is obtained by grouping the terms  $n$  and  $-n$ . Now, remark that  $d(\alpha)$  and  $\tilde{K}(q^\alpha)$  depend only of  $\alpha$  modulo  $2r$  so  $N(K(\alpha \pm 2r)) = \left(\frac{\theta_{\alpha \pm 2r}}{\theta_\alpha}\right)^f N(K(\alpha)) = q^{\pm 2rf\alpha} N(K(\alpha))$ . Using this, the sum of the  $n^{\text{th}}$  and the  $(r-n)^{\text{th}}$  terms of the previous sum is

$$\begin{aligned} &\{2n\} (N(K(\varepsilon - 2n))(1 - q^{2rf(\varepsilon - 2n)}) + N(K(\varepsilon + 2n))(1 - q^{-2rf(\varepsilon + 2n)})) \\ &= \{2n\} \{rf\varepsilon\} (q^{-rf\varepsilon} N(K(\varepsilon + 2n)) - q^{rf\varepsilon} N(K(\varepsilon - 2n))). \end{aligned}$$

But  $\lim_{\varepsilon \rightarrow 0} \{rf\varepsilon\} d(\varepsilon \pm 2n) = \lim_{\varepsilon \rightarrow 0} r \frac{rf\varepsilon}{\{r\varepsilon\}} \{\varepsilon \pm 2n\} = \pm rf\{2n\}$ . So,

$$S = \frac{rf\theta_0^f}{2\{1\}} \sum_{n=2, n \text{ even}}^{r-1} q^{2fn^2} \{2n\}^2 (\tilde{K}(q^{2n}) + \tilde{K}(q^{-2n}))$$

Thus, the proposition follows. 2.11

To show that  $N^0$  is non-trivial we will prove that its value for the Poincaré sphere  $P$  is not 1; thus distinguishing it from  $S^3$ . Let  $K$  be a trefoil with framing  $+1$ . Using the notation of

Subsection 2.3,  $K$  is equal to  $K_3^{-2}$  with the color  $\bar{0} \in \mathbb{C}/2\mathbb{Z}$ . Then for  $r = 5$  (and  $q = \exp(\frac{i\pi}{5})$ ), using Formula (9) and Equation (10) one obtains:

$$\tilde{K}(q^\alpha) = q[3\alpha] + q^{-1}[5\alpha] + q^2[9\alpha] \text{ where } [x] = \frac{\{x\}}{\{1\}}.$$

Then Proposition 2.11 implies

$$\mathbf{N}^0(P, \emptyset, 0) = \frac{5q^{-8}}{\{1\}\Delta_+} \left( q^8 \{4\}^2 \tilde{K}(q^4) + q^{32} \{8\}^2 \tilde{K}(q^8) \right) = -(q^2 + 1)^2 \neq 1 = \mathbf{N}^0(S^3, \emptyset, 0).$$

**Question 2.12.** *If  $M$  is a closed oriented 3-manifold, is there a relation between  $\mathbf{N}^0(M, \emptyset, 0)$  and the Reshetikhin-Turaev-Witten invariant of  $M$ ?*

**2.5. A holomorphic function on cohomology.** Let  $M$  be a 3-manifold and let  $T$  be a (possibly empty) framed trivalent graph in  $M$ . By fixing a basis of  $H_1(M \setminus T; \mathbb{Z})$  we have  $H_1(M \setminus T, \mathbb{Z}) = \mathbb{Z}^b \oplus \mathbb{Z}/n_i\mathbb{Z}$  for some positive integers  $b, n_i$ . Suppose also that  $b \geq 1$ ; then  $H^1(M \setminus T; \mathbb{C}/2\mathbb{Z})$  is isomorphic to  $(\mathbb{C}/2\mathbb{Z})^b \times \mathbb{Z}/n_i\mathbb{Z}$  where the isomorphism maps a class  $\omega$  to the list of its values on the elements of the basis. Note that if an element  $x$  has order  $n$  then  $\omega(x)$  is an integer multiple of  $\frac{2}{n} \in \mathbb{C}/2\mathbb{Z}$ .

Let  $(M, T, \omega)$  be a compatible triple. Let  $L$  be a link in  $S^3$  which represents a computable presentation of  $(M, T, \omega)$ . Let  $n_L$  be the number of components of  $L$  and  $L_i$  be the  $i^{\text{th}}$  component of  $L$ . For each  $i \in \{1, \dots, n_L\}$ , fix a complex number  $\alpha_i$  such that  $\overline{\alpha_i} = g_\omega(L_i)$ . By definition

$$\mathbf{N}(M, T, \omega) = \frac{1}{\Delta_+^p \Delta_-^s} \sum_{(k_i) \in (H_r)^{n_L}} \left( \prod_{i=1}^{n_L} d(\alpha_i + k_i) \right) \mathbf{N}(L^{(k_i)} \cup T)$$

where  $L^{(k_i)}$  is the link  $L$  such that  $L_i$  is colored by  $\alpha_i + k_i$ . Since  $L$  provides a computable presentation then  $\alpha_i \in \mathbb{C} \setminus \mathbb{Z}$  and so  $d(\alpha_i + k_i)$  is holomorphic in  $\alpha_i$ . Moreover, by Remark 1.2 the invariants  $\mathbf{N}(L^{(k_i)} \cup T)$  is holomorphic in the colorings. Therefore, when  $\mathbf{N}(M, T, \omega)$  is defined it is holomorphic in  $\omega$ . Similarly, one can show that  $\mathbf{N}(M, T, \omega)$  is holomorphic in  $\omega$  when defined using an  $H$ -stabilization.

**2.6. Behavior under connected sum.** We say that a compatible triple  $(M, T, \omega)$  is *generic* if  $T \neq \emptyset$  or if  $\omega$  is not integral. Notice that  $\mathbf{N}(M, T, \omega)$  is defined only if  $(M, T, \omega)$  is generic.

Let  $(M_1, T_1, \omega_1)$  and  $(M_2, T_2, \omega_2)$  be compatible triples. Recall the definition of the connected sum defined by Equation (5). We have the following three cases with regards to the connected sum:

- Case 1. Both  $(M_1, T_1, \omega_1)$  and  $(M_2, T_2, \omega_2)$  are generic. Then there exists a computable surgery presentation  $L_i$  of  $(M_i, T_i, \omega_i)$  (or a stabilization of  $(M_i, T_i, \omega_i)$ ), for  $i = 1, 2$ . Clearly,  $(M_1, T_1, \omega_1) \# (M_2, T_2, \omega_2)$  can be presented as surgery over  $L_1 \sqcup L_2$  which is a split link and thus  $\mathbf{N}((M_1, T_1, \omega_1) \# (M_2, T_2, \omega_2)) = 0$ .
- Case 2. Exactly one of the triples  $(M_1, T_1, \omega_1)$  or  $(M_2, T_2, \omega_2)$  is generic. Suppose  $(M_1, T_1, \omega_1)$  is not generic and then by Proposition 3.14 we have:

$$\mathbf{N}((M_1, T_1, \omega_1) \# (M_2, T_2, \omega_2)) = \mathbf{N}^0(M_1, T_1, \omega_1) \mathbf{N}(M_2, T_2, \omega_2).$$

- Case 3. Neither  $(M_1, T_1, \omega_1)$  nor  $(M_2, T_2, \omega_2)$  are generic. Then  $(M_1, T_1, \omega_1) \# (M_2, T_2, \omega_2)$  is not generic and Proposition 3.15 implies:

$$\mathbf{N}^0((M_1, T_1, \omega_1) \# (M_2, T_2, \omega_2)) = \mathbf{N}^0(M_1, T_1, \omega_1) \mathbf{N}^0(M_2, T_2, \omega_2).$$



**2.7. Studying the self-diffeomorphisms of a rational homology sphere.** Let  $M$  be a rational homology sphere. Let  $c_1, \dots, c_n$  and  $d_1, \dots, d_n$  be two distinct minimal sets of generators of  $H_1(M; \mathbb{Z})$ . Suppose that one is given  $\omega, \omega' \in H^1(M; \mathbb{C}/2\mathbb{Z})$  such that  $\omega'(c_i) = \omega(d_i)$ . If  $N(M, \emptyset, \omega') \neq N(M, \emptyset, \omega)$  then there exists no positive self-diffeomorphism  $\phi : M \rightarrow M$  such that  $\phi_*(c_i) = d_i$ . In particular, one can apply this argument to distinguish generators of  $H_1(M; \mathbb{Z})$ . For example, consider  $H_1(L(5, 1); \mathbb{Z})$ . Present  $L(5, 1)$  as the surgery over a 5-framed unknot in  $S^3$ ; a generator  $c_1$  is represented by the meridian of the unknot, and another,  $d_1$ , by its double. Using Formula (6) with  $r = 3$  and a computer one can see that

$$N(L(5, 1), \emptyset, \omega) \neq N(L(5, 1), \emptyset, \omega')$$

where  $\omega(c_1) = \omega'(d_1) = \frac{2}{5} \in \mathbb{C}/2\mathbb{Z}$ .

**2.8. Conjugation versus change of orientation.** Let  $T \subset S^3$  be a framed, oriented graph with set of vertices  $\mathcal{V}$  and set of edges  $\mathcal{E}$ . Fix  $\delta : \mathcal{V} \rightarrow H_r$  and assume that  $T$  has a  $\mathbb{C} \setminus X_r$ -coloring with boundary  $\delta$  (see Remark 1.2). Let  $R_T = \mathbb{Q}(q)(\{q^{\frac{e_i}{4}}, q^{\frac{e_j e_k}{4}} | e_i, e_j, e_k \in \mathcal{E}\})$  and let  $i : R_T \rightarrow R_T$  be the  $\mathbb{Q}$ -linear isomorphism defined by  $i(q) = q^{-1}$ ,  $i(q^{\frac{e_i}{4}}) = (q^{\frac{e_i}{4}})^{-1}$ , and  $i(q^{\frac{e_j e_k}{4}}) = (q^{\frac{e_j e_k}{4}})^{-1}$  for all  $e_i, e_j, e_k \in \mathcal{E}$ . By the axioms of Section 1.2,  $N(T)$  can be seen as the evaluation of an element  $N_\delta(T) \in R_T$  via the map  $\text{ev} : R_T \rightarrow \mathbb{C}$  defined by  $\text{ev}(q) = \exp(\frac{i\pi}{r})$ ,  $\text{ev}(q^{\frac{e_i}{4}}) = \exp(\frac{\pi i \alpha_i}{4r})$  and  $\text{ev}(q^{\frac{e_j e_k}{4}}) = \exp(\frac{\pi i \alpha_j \alpha_k}{4r})$  where  $\alpha_i \in \mathbb{C}$  denotes the color of  $e_i$ .

**Lemma 2.13.** *Let  $T \subset S^3$  be a framed, oriented, colored graph. Let  $T^*$  be the mirror image of  $T$  equipped with the coloring obtained by conjugating all the colors of the edges of  $T$  and with the framing obtained by taking the mirror image of the framing of  $T$ . Then  $N(T^*) = \overline{\text{ev}(i(N_\delta(T)))} = \overline{N(T)} \in \mathbb{C}$ .*

*Proof.* First remark that if  $T$  is planar, then  $\text{ev}(i(N_\delta(T))) = N(T)$  (and so the second equality is proved): indeed using the axioms of Section 1.2  $N_\delta(T)$  can be expressed as a sum of products of the values of  $6j$ -symbols, of the theta-graphs and of unknots, each of which is expressed in terms of quantum binomials and are thus fixed by  $i$  because  $i(\{x\}) = -\{x\}$ ,  $\forall x \in \mathbb{C}$  and each binomial contains an even number of such terms. Moreover for planar  $T$  if  $\overline{T}$  is the mirror image of  $T$  equipped with the same coloring as  $T$ , then  $N(\overline{T}) = N(T)$ : to show this one may check that the value of a tetrahedron and its mirror image are equal and this is proved by applying four times the Axiom (N g); then expressing again  $N(T)$  and  $N(\overline{T})$  as sums of products of values of  $6j$ -symbols, theta-graphs and unknots one gets that  $N(\overline{T}) = N(T)$ . So to prove the first statement for planar  $T$ , it is sufficient to remark that for any  $x \in \mathbb{C}$  it holds  $\overline{(q^x)} = q^{-\overline{x}}$  then  $i(\{x\}) = -i(\{\overline{x}\})$  and since  $q$ -binomials contain an even number of factors of the form  $i(\{x\})$  it holds  $i\left(\left[\begin{smallmatrix} x \\ a \end{smallmatrix}\right]\right) = i\left(\left[\begin{smallmatrix} \overline{x} \\ \overline{a} \end{smallmatrix}\right]\right)$ . Thus once again since the invariant of a planar graph can be expressed as sum of products of  $q$ -binomials the claim follows and so  $\overline{N(T)} = \overline{N(\overline{T})} = N(T^*)$ . For non planar  $T$ , given a diagram of it, using axioms (N i) and (N g) one can express  $N(T)$  as a sum of products of  $6j$ -symbols, theta graphs, unknots and some powers of  $q$  whose exponent is a degree 2-polynomial in the colors of  $T$ . Then one is left to check that switching all the crossings in a diagram of  $T$  has the same effect on  $N_\delta(T)$  as  $i$ : this is checked directly by the axioms (N f) and (N g) because changing a factor from  $q^{\frac{\alpha^2 - (r-1)^2}{4}}$  to  $q^{-\frac{\alpha^2 - (r-1)^2}{4}}$  is exactly applying  $\text{ev}(i(\cdot))$ . 2.13

Given a compatible triple  $(M, T, \omega)$  let  $\overline{M}$  be  $M$  with the opposite orientation,  $-T$  be  $T$  with the opposite orientations on the edges,  $\overline{T}$  be  $T$  equipped with the conjugated coloring and  $\overline{\omega}$  be the complex conjugate of  $\omega$ ; observe that  $(M, -T, -\omega)$ ,  $(\overline{M}, -T, \omega)$  and  $(M, \overline{T}, \overline{\omega})$  are all

compatible triples and hence also  $(\overline{M}, \overline{T}, -\overline{\omega})$  is. If  $(M, T, \omega)$  is presented as an integral surgery over a framed link  $L$  in  $S^3$  (with  $T \subset S^3 \setminus L$ ), then, letting  $L^* \cup T^*$  be the mirror image of  $L \cup T$  (as in the statement of Lemma 2.13), it can be checked that  $(\overline{M}, \overline{T}, -\overline{\omega})$  can be presented as surgery over  $L^*$  (with the framing obtained by taking the mirror image of the framing of  $L$ ) and with  $\overline{T}$  being presented as  $T^*$ . By Lemma 2.13 and the definitions of  $\mathbf{N}$  and of  $\mathbf{d}$  it holds  $\mathbf{N}(\overline{M}, T^*, -\overline{\omega}) = \mathbf{N}(\overline{M}, T, \omega)$ .

### 3. THE GENERAL CONSTRUCTION OF THE 3-MANIFOLD INVARIANT

In this section we give a general categorical construction of an invariant of a colored graph in a compact connected oriented 3-manifold and a cohomology class. In Section 5 we will show that the invariant  $\mathbf{N}$  define in Subsection 1.4 is a special case of the invariant defined in this section.

**3.1. Ribbon Ab-categories.** We describe the concept of a ribbon Ab-category (for details see [34]). A *tensor category*  $\mathcal{C}$  is a category equipped with a covariant bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  called the tensor product, a unit object  $\mathbb{I}$ , an associativity constraint, and left and right unit constraints such that the Triangle and Pentagon Axioms hold. When the associativity constraint and the left and right unit constraints are all identities we say the category  $\mathcal{C}$  is a *strict* tensor category. By Mac Lane's coherence theorem any tensor category is equivalent to a strict tensor category.

A tensor category  $\mathcal{C}$  is said to be an *Ab-category* if for any pair of objects  $V, W$  of  $\mathcal{C}$  the set of morphism  $\text{Hom}(V, W)$  is an additive abelian group and the composition and tensor product of morphisms are bilinear.

Let  $\mathcal{C}$  be a (strict) ribbon Ab-category, i.e. a (strict) tensor Ab-category with duality, a braiding and a twist. Composition of morphisms induces a commutative ring structure on  $\text{End}(\mathbb{I})$ . This ring is called the *ground ring* of  $\mathcal{C}$  and denoted by  $\mathbb{K}$ . For any pair of objects  $V, W$  of  $\mathcal{C}$  the abelian group  $\text{Hom}(V, W)$  becomes a left  $\mathbb{K}$ -module where the action is defined by  $kf = k \otimes f$  where  $k \in \mathbb{K}$  and  $f \in \text{Hom}(V, W)$ . An object  $V$  of  $\mathcal{C}$  is *simple* if  $\text{End}(V) = \mathbb{K} \text{Id}_V$ . We denote the braiding in  $\mathcal{C}$  by  $c_{V, W} : V \otimes W \rightarrow W \otimes V$ , the twist in  $\mathcal{C}$  by  $\theta_V : V \rightarrow V$  and duality morphisms in  $\mathcal{C}$  by

$$b_V : \mathbb{I} \rightarrow V \otimes V^*, \quad b'_V : \mathbb{I} \rightarrow V^* \otimes V, \quad d_V : V^* \otimes V \rightarrow \mathbb{I}, \quad d'_V : V \otimes V^* \rightarrow \mathbb{I}.$$

An object  $V$  of  $\mathcal{C}$  is a direct sum of the objects  $V_1, \dots, V_n$  if there exist morphisms  $\alpha_i : V_i \rightarrow V$  and  $\beta_i : V \rightarrow V_i$  for  $i = 1 \dots n$  with  $\beta_j \alpha_i = \delta_i^j \text{Id}_{V_i}$  and  $\text{Id}_V = \sum_{i=1}^n \alpha_i \beta_i$ . We say that a full subcategory  $\mathcal{D}$  of  $\mathcal{C}$  is *semi-simple* if every object of  $\mathcal{D}$  is a direct sum of finitely many simple objects in  $\mathcal{D}$  and any two non-isomorphic simple objects  $V$  and  $W$  of  $\mathcal{D}$  have the property  $\text{Hom}_{\mathcal{C}}(V, W) = 0$ .

**3.2. Ribbon graphs.** Here we recall the notion of the category of  $\mathcal{C}$ -colored ribbon graphs and its associated ribbon functor (for details see [34]). A morphism  $f : V_1 \otimes \dots \otimes V_n \rightarrow W_1 \otimes \dots \otimes W_m$  in  $\mathcal{C}$  can be represented by a box and arrows as in Figure 2. Here the plane of the picture is

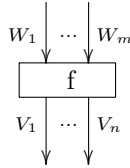


FIGURE 2.

oriented counterclockwise, and this orientation determines the numbering of the arrows attached to the bottom and top of the box. More generally, we allow such boxes with some arrows directed up and some arrows directed down. For example, if all the bottom arrows in the above picture are redirected upward, then the box represents a morphism  $V_1^* \otimes \cdots \otimes V_n^* \rightarrow W_1 \otimes \cdots \otimes W_m$ . The boxes as above are called *coupons*. Each coupon has distinguished bottom and top sides and all incoming and outgoing arrows can be attached only to these sides.

Let  $M$  be an oriented 3-manifold. A *ribbon graph* in  $M$  is an oriented compact surface in  $M$  which decomposed into elementary pieces: bands, annuli, and coupons (see [34]). A  $\mathcal{C}$ -*ribbon graph* in  $M$  is a ribbon graph whose bands and annuli are colored by objects of  $\mathcal{C}$  and whose coupons are colored with morphisms of  $\mathcal{C}$ . Recall the definition of a framed colored graph given in Subsection 1.1. A  $\mathcal{C}$ -ribbon graph in  $\mathbb{R}^2 \times [0, 1]$  is a  $Ob(\mathcal{C})$ -colored framed oriented graph such that all the vertices lying in  $\text{Int}(\mathbb{R}^2 \times [0, 1])$  are not univalent and are thickened to coupons colored by morphisms of  $\mathcal{C}$ . Let  $A$  be a set of simple objects of  $\mathcal{C}$ . By a  $A$ -*graph* in  $M$  we mean a  $\mathcal{C}$ -ribbon graph in  $M$  where at least one band or annuli is colored by an element of  $A$ . By an  $A$ -*graph*, we mean a closed  $A$ -graph in  $S^3$  or in  $B^3$ .

Next we recall the category of  $\mathcal{C}$ -colored ribbon graphs  $Rib_{\mathcal{C}}$  (for more details see [34] Chapter I). The objects of  $Rib_{\mathcal{C}}$  are sequences of pairs  $(V, \epsilon)$ , where  $V \in Ob(\mathcal{C})$  and  $\epsilon = \pm$  determines the orientation of the corresponding edge. The morphisms of  $Rib_{\mathcal{C}}$  are isotopy classes of  $\mathcal{C}$ -colored ribbon graphs in  $\mathbb{R}^2 \times [0, 1]$  and their formal linear combinations with coefficients in  $\mathbb{K}$ . From now on we write  $V$  for  $(V, +)$ .

Let  $F$  be the usual ribbon functor from  $Rib_{\mathcal{C}}$  to  $\mathcal{C}$  (see [34]). Let  $L$  be a  $A$ -graph and let  $V$  be the color of an edge of  $L$  belonging to  $A$ . Cutting this edge, we obtain a  $\mathcal{C}$ -colored  $(1, 1)$ -ribbon graph  $T_V$  whose closure is  $L$ . Since  $V$  is simple  $F(T_V) \in \text{End}_{\mathcal{C}}(V) = \mathbb{K} \text{Id}_V$ . Let  $\langle T_V \rangle \in \mathbb{K}$  be such that  $F(T_V) = \langle T_V \rangle \text{Id}_V$ . Let  $d : A \rightarrow \mathbb{K}^*$  be a map. We say  $(A, d)$  is an *ambidextrous pair* if, for all  $L$ ,

$$(11) \quad F'(L) = d(V) \langle T_V \rangle \in \mathbb{K}$$

is independent of the choice of the edge to be cut and yields a well defined invariant of  $L$ .

**3.3. Relative  $G$ -modular categories.** In this subsection we give the main new categorical notion of this paper. If  $G$  is a commutative group (in this section we use a multiplicative notation), a  $G$ -*grading* in  $\mathcal{C}$  is a family  $\{\mathcal{C}_g\}_{g \in G}$  of full subcategories of  $\mathcal{C}$  such that:

- (1) If  $V \in \mathcal{C}_g, V' \in \mathcal{C}_{g'}$  then  $V \otimes V' \in \mathcal{C}_{gg'}$ .
- (2) If  $V \in \mathcal{C}_g$  then  $V^* \in \mathcal{C}_{g^{-1}}$ .
- (3) If  $V \in \mathcal{C}_g, V' \in \mathcal{C}_{g'}$  and  $g \neq g'$  then  $\text{Hom}_{\mathcal{C}}(V, V') = 0$ .

A set of objects of  $\mathcal{C}$  is said to be *commutative* if for any pair  $(V, W)$  of objects in this set we have  $c_{V, W} \circ c_{W, V} = \text{Id}_{W \otimes V}$  and  $\theta_V = \text{Id}_V$ . Let  $Z$  be a commutative group with additive notation. A *realization* of  $Z$  in  $\mathcal{C}$  is a commutative set of object  $\{\varepsilon^t\}_{t \in Z}$  such that  $\varepsilon^0 = \mathbb{I}$ ,  $\text{qdim}(\varepsilon^t) = 1$  and  $\varepsilon^t \otimes \varepsilon^{t'} = \varepsilon^{t+t'}$  for all  $t, t' \in Z$ .

**Lemma 3.1.** *Let  $\{\varepsilon^t\}_{t \in Z}$  be a realization of  $Z$  in  $\mathcal{C}$ . If  $t \in Z$  then  $\varepsilon^t$  is simple. Also, for any objects  $V, W \in \mathcal{C}$ , the map  $\text{Hom}_{\mathcal{C}}(V, W) \rightarrow \text{Hom}_{\mathcal{C}}(V \otimes \varepsilon^t, W \otimes \varepsilon^t)$  given by  $f \mapsto f \otimes \text{Id}_{\varepsilon^t}$  is an isomorphism. In particular, if  $V$  is simple then  $V \otimes \varepsilon^t$  is also simple.*

*Proof.* We have that for any  $t \in Z$ ,  $\text{Id}_{\varepsilon^t} \otimes \text{Id}_{\varepsilon^{-t}} = \text{Id}_{\mathbb{I}}$ . It follows that the map  $\text{Hom}_{\mathcal{C}}(V, W) \rightarrow \text{Hom}_{\mathcal{C}}(V \otimes \varepsilon^t, W \otimes \varepsilon^t)$  which sends  $f$  to  $f \otimes \text{Id}_{\varepsilon^t}$  has an inverse map given by  $f \mapsto f \otimes \text{Id}_{\varepsilon^{-t}}$ . In particular, if  $V$  is simple then  $\text{End}_{\mathcal{C}}(V) = \mathbb{K} \cdot \text{Id}_V$  implying that  $\text{End}_{\mathcal{C}}(V \otimes \varepsilon^t) = \mathbb{K} \cdot \text{Id}_{V \otimes \varepsilon^t}$ . Finally, taking  $V = \mathbb{I}$  we have that  $\varepsilon^t$  is simple. 3.1

$$F \left( \begin{array}{c} \text{diagram of a long Hopf link with a loop labeled } \Omega_g \text{ and a strand labeled } V \end{array} \right) = \Delta_- \text{Id}_V, \quad F \left( \begin{array}{c} \text{diagram of a long Hopf link with a loop labeled } \Omega_g \text{ and a strand labeled } V \end{array} \right) = \Delta_+ \text{Id}_V$$

FIGURE 3. Here  $V$  is in  $\mathcal{C}_g$  and  $\Omega_g$  is a formal linear combination of modules  $\sum_{U \in Y} d(U)U$  where  $Y \subset \text{Obj}(\mathcal{C}_g)$  is a finite set representing the simple  $\mathbb{Z}$ -orbits in  $\mathcal{C}_g$ .

A realization of  $\mathbb{Z}$  in  $\mathcal{C}$  induces an action of  $\mathbb{Z}$  on isomorphism classes of objects of  $\mathcal{C}$  by  $(t, V) \mapsto \varepsilon^t \otimes V \simeq V \otimes \varepsilon^t$  where the isomorphism here is given by the braiding. We say that  $\{\varepsilon^t\}_{t \in \mathbb{Z}}$  is a *free realization* of  $\mathbb{Z}$  in  $\mathcal{C}$  if this action is free. This means that for any  $t \in \mathbb{Z} \setminus \{0\}$  and for any object  $V$  of  $\mathcal{C}$ ,  $V \otimes \varepsilon^t \not\simeq V$ .

For a simple object  $V$ , we denote by  $\tilde{V}$  the set of isomorphism classes of the set of simple objects  $\{V \otimes \varepsilon^t | t \in \mathbb{Z}\}$ . We say that  $\tilde{V}$  is a *simple  $\mathbb{Z}$ -orbit*.

**Definition 3.2.** A ribbon category  $\mathcal{C}$  is  *$G$ -modular relative to  $\overline{X}$  with modified dimension  $d$  and periodicity group  $\mathbb{Z}$*  if

- (1) the category  $\mathcal{C}$  has a  $G$ -grading  $\{\mathcal{C}_g\}_{g \in G}$ ,
- (2) there is a free realization  $\{\varepsilon^t\}_{t \in \mathbb{Z}}$  of the group  $\mathbb{Z}$  in  $\mathcal{C}_1$  (where  $1 \in G$  is the unit),
- (3) there is a bilinear pairing  $G \times \mathbb{Z} \rightarrow \mathbb{K}^*$ ,  $(g, t) \mapsto g^{\bullet t}$  such that for any object  $V$  of  $\mathcal{C}_g$  we have  $c_{V, \varepsilon^t} \circ c_{\varepsilon^t, V} = g^{\bullet t} \text{Id}_{\varepsilon^t \otimes V}$ , for all  $t \in \mathbb{Z}$ ,
- (4) there exists  $\overline{X} \subset G$  such that  $\overline{X}^{-1} = \overline{X}$  and  $G$  cannot be covered by a finite number of translated copies of  $\overline{X}$ , in other words, for any  $g_1, \dots, g_n \in G$ , we have  $\bigcup_{i=1}^n (g_i \overline{X}) \neq G$ ,
- (5) there is an ambidextrous pair  $(A, d)$  where  $A$  contains the set of simple objects of  $\mathcal{C}_g$  for all  $g \in G \setminus \overline{X}$ ,
- (6) for all  $g \in G \setminus \overline{X}$ , the category  $\mathcal{C}_g$  is semi-simple and its simple objects form a union of finitely many simple  $\mathbb{Z}$ -orbits,
- (7) there exists an element  $g \in G \setminus \overline{X}$  and an object  $V \in \mathcal{C}_g$  such that the scalar  $\Delta_+$  given in Figure 3 is non-zero (similarly, there exists  $g \in G \setminus \overline{X}$  and  $V \in \mathcal{C}_g$  such that  $\Delta_- \neq 0$ ),
- (8)  $F(H(V, W)) \neq 0$ , for all  $V, W \in A$ , where  $H(V, W)$  is the long Hopf link whose long edge is colored  $V$  and circle component is colored with  $W$ .

Remark that the bilinearity of the pairing  $G \times \mathbb{Z} \rightarrow \mathbb{K}^*$  means that  $g^{\bullet t+t'} = g^{\bullet t} g^{\bullet t'}$  and  $(gh)^{\bullet t} = g^{\bullet t} h^{\bullet t}$ . We can illustrate Condition (3) with the following skein relation:

$$(12) \quad F \left( \begin{array}{c} \text{diagram of a crossing with strands labeled } \varepsilon^t \text{ and } V \end{array} \right) = g^{\bullet t} F \left( \begin{array}{c} \text{diagram of a crossing with strands labeled } \varepsilon^t \text{ and } V \end{array} \right) \text{ for all } V \in \mathcal{C}_g.$$

**Notation.** If  $\mathcal{C}$  is a category satisfying Definition 3.2 then we say  $\mathcal{C}$  is a *relative  $G$ -modular category*. For such a category let  $\{\alpha\}$  be a set indexing simple objects of  $A$ . Let  $V \in A$  and let  $\alpha$  be the corresponding indexing element. We will denote  $\overline{\alpha}$  as the unique element of  $G$  such that  $V \in \mathcal{C}_{\overline{\alpha}}$ . Also, to simplify notation we will identify  $\alpha$  with  $V$  and write  $\alpha \in A$ .

**Remark 3.3.** Conditions (1) and (2) of Definition 3.2 are not very restrictive once one has a free realization of  $\mathbb{Z}$  in  $\mathcal{C}$ : The long Hopf link given by a straight strand colored by  $V$  and its meridian colored by  $\varepsilon^t$  is sent by  $F$  to a central isomorphism  $\phi_t(V) \in \text{End}_{\mathcal{C}}(V)$ . For  $g \in G' = \text{Hom}_{\mathcal{C}}(\mathbb{Z}, \mathbb{K}^*)$ , let  $\mathcal{C}'_g$  be the full subcategory of  $\mathcal{C}$  formed by objects  $V$  such that  $\phi_t(V) = g(t) \text{Id}_V$ , for all  $t \in \mathbb{Z}$ . If  $V$  is a simple object of  $\mathcal{C}$  then  $\phi_t(V)$  is a non-zero scalar and so  $V$  belongs to

some  $\mathcal{C}'_g$ . Furthermore, one can easily prove that  $(\mathcal{C}'_g)_{g \in G'}$  is a grading in  $\mathcal{C}$  and  $\varepsilon^t \in \mathcal{C}'_1$ , for all  $t \in \mathbb{Z}$ .

**Remark 3.4.** Condition (8) holds in all the examples of this paper. In general this assumption is not true, however the graph  $H$  can be exchanged with any ribbon graph which does not vanish when colored by elements of  $\mathbf{A}$ . For such an exchange, in what follows, the process of  $H$ -stabilization below should be replaced by the connected sum with the new ribbon graph.

**3.4. Main results.** Here we give the general definition of the invariants this paper. Recall the notation and definitions of Subsection 1.3. Let  $\mathcal{C}$  be a relative  $G$ -modular category relative to  $\overline{X}$  with modified dimension  $\mathbf{d}$  and periodicity group  $\mathbb{Z}$ . A formal linear combination of objects of  $\mathcal{C}$  is a *homogenous  $\mathcal{C}$ -color* of degree  $g \in G$  if all appearing objects belong to the same  $\mathcal{C}_g$ . We say a ribbon graph  $T$  has a *homogenous  $\mathcal{C}$ -coloring* if each edge  $e$  of  $T$  is colored by a homogenous  $\mathcal{C}$ -color of degree  $g_e \in G$  for some 1-cycle  $\sum_e g_e[e] \in H_1(T, G)$  which is called *the  $G$ -coloring* of  $T$ .

**Definition 3.5.** Let  $M$  be a compact connected oriented 3-manifold,  $T$  a  $\mathcal{C}$ -colored ribbon graph in  $M$  and  $\omega \in H^1(M \setminus T, G)$ .

- (1) We say that  $(M, T, \omega)$  is a *compatible triple* if  $T$  has a homogenous  $\mathcal{C}$ -coloring given by  $\Phi(\omega) \in H_1(T, G)$ .
- (2) A compatible triple is  *$T$ -admissible* if there exists an edge of  $T$  colored by  $\alpha \in \mathbf{A}$ .
- (3) A surgery presentation via  $L \subset S^3$  for a compatible triple  $(M, T, \omega)$  is *computable* if one of the two following conditions holds:
  - (a)  $L \neq \emptyset$  and  $g_\omega(L_i) \in G \setminus \overline{X}$  for all  $L_i$  or
  - (b)  $L = \emptyset$  and there exists an edge of  $T$  colored by  $\alpha \in \mathbf{A}$ .

From now on we assume that  $(M, T, \omega)$  is a compatible triple.

**Definition 3.6.** The formal linear combination  $\Omega_g = \sum_i \mathbf{d}(V_i)V_i$  is a *Kirby color of degree  $g \in G$*  if the isomorphism classes of the  $\{V_i\}_i$  are in one to one correspondence with the simple  $\mathbb{Z}$ -orbits of  $\mathcal{C}_g$ .

**Theorem 3.7.** *If  $L$  is a link which gives rise to a computable surgery presentation of  $(M, T, \omega)$  then*

$$N(M, T, \omega) = \frac{F'(L \cup T)}{\Delta_+^p \Delta_-^s}$$

*is a well defined topological invariant (i.e. depends only of the diffeomorphism class of the triple  $(M, T, \omega)$ ), where  $(p, s)$  is the signature of the linking matrix of the surgery link  $L$  and each component  $L_i$  is colored by a Kirby color  $\Omega_{g_\omega(L_i)}$ .*

The proof of Theorem 3.7 will be given in Section 4.

**Proposition 3.8.** *Let  $(M, T, \omega)$  be a compatible triple and consider  $\omega$  as a map on  $H_1(M \setminus T, \mathbb{Z})$  with values in  $G$ . Suppose that  $\omega$  takes a value  $g \in G$  such that for each  $x \in \overline{X}$  there exists an  $n(x) \in \mathbb{Z}$  such that  $xg^{n(x)} \notin \overline{X}$ . Then there exists a computable surgery presentation of  $(M, T, \omega)$ .*

The proof of Proposition 3.8 will be given in Section 4.

**Definition 3.9.** [ $H$ -Stabilization] Let  $H(\alpha, \beta)$  be a long Hopf link in  $\mathbb{R}^3$  whose circle component is colored by  $\alpha \in \mathbf{A}$  and whose long component is colored by  $\beta \in \mathbf{A}$ . Let  $(M, T, \omega)$  be a  $T$ -admissible triple,  $e$  be an edge of  $T$  colored by  $\beta \in \mathbf{A}$ , and  $m$  be the meridian of  $e$ . A  *$H$ -stabilization* of  $(M, T, \omega)$  along  $e$  is a compatible triple  $(M, T_H, \omega_H)$  where:

- $T_H = T \cup m$ , and  $m$  is colored by  $\alpha \in \mathbf{A}$ ,
- $\omega_H$  is the unique element of  $H^1(M \setminus (T \cup m); G)$  such that  $\omega_H(m) = \bar{\alpha}$  and  $(\omega_H)|_{M \setminus (T \cup D)} = \omega$  where  $D$  is a disc bounded by  $m$  which  $e$  intersects once.

**Theorem 3.10.** *If  $(M, T, \omega)$  is  $T$ -admissible then there exists a  $H$ -stabilization of  $(M, T, \omega)$  admitting a computable presentation. Let  $(M, T_H, \omega_H)$  be such a  $H$ -stabilization and let  $L$  be a link which gives rise to a computable surgery presentation of  $(M, T_H, \omega_H)$  then*

$$\mathbf{N}(M, T, \omega) = \frac{F'(L \cup T_H)}{\langle H \rangle \Delta_+^p \Delta_-^s}$$

is a well defined topological invariant (i.e. depends only of the diffeomorphism class of the triple  $(M, T, \omega)$ ), where  $(p, s)$  is the signature of the linking matrix of the surgery link  $L$ , each component  $L_i$  is colored by a Kirby color  $\Omega_{g_\omega(L_i)}$ ,  $H = H(\alpha, \beta)$  is the long Hopf-link used in the stabilization and  $\langle H \rangle$  is defined by the equality  $F(H) = \langle H \rangle \text{Id}_\beta$ . Moreover, if  $(M, T, \omega)$  has a computable presentation, then the invariant of this theorem is equal to the invariant of Theorem 3.7.

The proof of Theorem 3.10 will be given in Section 4. Next we define another invariant which can be non-zero when  $\mathbf{N}$  is zero. Before we do this we need the following result.

Let  $T$  and  $T'$  be  $\mathcal{C}$ -colored ribbon graphs and let  $e \subset T, e' \subset T'$  be edges colored by  $\alpha \in \mathbf{A}$ . Let  $T \#_{(e, e')} T'$  be the connected sum of  $T$  and  $T'$  along  $e$  and  $e'$ , then

$$(13) \quad F'(T \#_{(e, e')} T') = \frac{1}{d(\alpha)} F'(T) F'(T').$$

Let  $T$  be a  $\mathcal{C}$ -colored ribbon graph with an edge  $e$  colored by  $\alpha \in \mathbf{A}$ . Let  $m$  be a meridian of  $e$  colored with  $\beta \in \mathbf{A}$ . If  $T' = T \cup m$  then

$$(14) \quad F'(T') = F'(T) \langle H \rangle$$

where  $H$  is the long Hopf link whose long edge is colored  $\alpha$  and circle component is colored with  $\beta$  and  $\langle H \rangle$  is defined by the equality  $F(H) = \langle H \rangle \text{Id}_\alpha$ .

**Proposition 3.11.** *If  $(M, T, \omega)$  is  $T$ -admissible and  $(M', T', \omega')$  is  $T'$ -admissible and  $e \subset T, e' \subset T'$  are edges colored by  $\beta \in \mathbf{A}$ , then*

$$\mathbf{N}(M \# M', T \#_{(e, e')} T', \omega \#_{(e, e')} \omega') = \frac{1}{d(\beta)} \mathbf{N}(M, T, \omega) \mathbf{N}(M', T', \omega')$$

where the connected sum is taken over balls intersecting  $T$  and  $T'$  along  $e$  and  $e'$ ,  $T \#_{(e, e')} T'$  is the connected sum of  $T$  and  $T'$  along  $e$  and  $e'$  and  $\omega \#_{(e, e')} \omega'$  is the cohomology class acting as  $\omega$  and  $\omega'$  on the images through the natural maps from  $H_1(M \setminus T, \mathbb{Z})$  and  $H_1(M' \setminus T', \mathbb{Z})$  in  $H_1(M \# M' \setminus T \#_{(e, e')} T', \mathbb{Z})$ .

*Proof.* Let  $(M, T_H, \omega_H)$  and  $(M', T_{H'}, \omega'_{H'})$  be  $H$ -stabilizations of  $(M, T, \omega)$  and  $(M', T', \omega')$  along  $e$  and  $e'$ , respectively (here  $H = H(\alpha, \beta)$  and  $H' = H(\alpha', \beta)$  are long Hopf links and  $\alpha, \alpha' \in \mathbf{A}$ ). By definition

$$\mathbf{N}(M, T, \omega) = \frac{F'(L \cup T_H)}{\langle H \rangle \Delta_+^p \Delta_-^s} \quad \text{and} \quad \mathbf{N}(M, T', \omega') = \frac{F'(L' \cup T_{H'})}{\langle H' \rangle \Delta_+^{p'} \Delta_-^{s'}}$$

where  $L$  and  $L'$  are links colored as in Theorem 3.10 and  $(p, s)$  (resp.  $(p', s')$ ) is the signature of the linking matrix of  $L$  (resp.  $L'$ ).

Using Equation (13) we have

$$\frac{F'(L \cup T_H)}{\langle H \rangle \Delta_+^p \Delta_-^s} \cdot \frac{F'(L' \cup T_{H'})}{\langle H' \rangle \Delta_+^{p'} \Delta_-^{s'}} = \frac{d(\beta) F'((L \cup T_H) \#_{(e, e')} (L' \cup T_{H'}))}{\langle H \rangle \langle H' \rangle \Delta_+^{p+p'} \Delta_-^{s+s'}}.$$

Now

$$\begin{aligned} F'((L \cup T_H) \#_{(e,e')}(L' \cup T'_{H'})) &= \langle H' \rangle F'((L \cup T_H) \#_{(e,e')}(L' \cup T')) \\ &= \langle H' \rangle F'((L \cup L') \cup (T \#_{(e,e')} T')_H). \end{aligned}$$

where the first equality follows from Equation (14) and the second from the fact that the two graphs in  $F'$  are isotopic. By definition

$$\mathbf{N}(M \# M', T \#_{(e,e')} T', \omega \#_{(e,e')} \omega') = \frac{F'((L \cup L') \cup (T \#_{(e,e')} T')_H)}{\langle H \rangle \Delta_+^{p+p'} \Delta_-^{s+s'}}$$

thus the result follows from the last two equations. 3.11

Let  $(M, T, \omega)$  be compatible triple. Let  $u_\alpha$  be an unknot in  $S^3$  colored by  $\alpha \in \mathbf{A}$ . Let  $\omega_\alpha$  be the unique element of  $H^1(S^3 \setminus u_\alpha, G)$  such that  $(S^3, u_\alpha, \omega_\alpha)$  is a compatible triple. Recall the definition of the connected sum given by Equation (5).

**Theorem 3.12.** *Define*

$$\mathbf{N}^0(M, T, \omega) = \frac{\mathbf{N}((M, T, \omega) \# (S^3, u_\alpha, \omega_\alpha))}{\mathbf{d}(\alpha)}.$$

*Then  $\mathbf{N}^0(M, T, \omega)$  is a well defined topological invariant (i.e. depends only of the diffeomorphism class of the compatible triple  $(M, T, \omega)$ ).*

*Proof.* We must show the definition of  $\mathbf{N}^0$  does not depend on the color  $\alpha \in \mathbf{A}$ . Let  $\alpha, \beta \in \mathbf{A}$ . Let  $u_\beta$  be the unknot with color  $\beta$  and edge  $e_\beta$ . Let  $\omega_\beta$  be the unique element of  $H^1(S^3 \setminus u_\beta, G)$  such that  $(S^3, u_\beta, \omega_\beta)$  is a compatible triple. Let  $H$  be the Hopf link whose edges  $e_1$  and  $e_2$  are colored with  $\alpha$  and  $\beta$ , respectively. Let  $\omega_{\alpha,\beta}$  be the unique element of  $H^1(S^3 \setminus H, G)$  such that  $(S^3, H, \omega_{\alpha,\beta})$  is a compatible triple. Consider the cohomology class  $\omega \sqcup \omega_\beta$  of  $(M, T, \omega) \# (S^3, u_\beta, \omega_\beta)$ . Then

$$\begin{aligned} \mathbf{N}((M, T, \omega) \# (S^3, H, \omega_{\alpha,\beta})) &= \mathbf{N}(M \# S^3, (T \sqcup u_\beta) \#_{(e_\beta, e_2)} H, (\omega \sqcup \omega_\beta) \#_{(e_\beta, e_2)} \omega_{\alpha,\beta}) \\ &= \frac{\mathbf{N}((M, T, \omega) \# (S^3, u_\beta, \omega_\beta)) \mathbf{N}(S^3, H, \omega_{\alpha,\beta})}{\mathbf{d}(\beta)} \end{aligned}$$

where the first equality from follows the fact that the two triples are diffeomorphic and the second equality comes from Proposition 3.11. Similarly,

$$\mathbf{N}((M, T, \omega) \# (S^3, H, \omega_{\alpha,\beta})) = \frac{\mathbf{N}((M, T, \omega) \# (S^3, u_\alpha, \omega_\alpha)) \mathbf{N}(S^3, H, \omega_{\alpha,\beta})}{\mathbf{d}(\alpha)}.$$

Thus, the theorem follows from the last two equations. 3.12

**Remark 3.13.** Any modular category satisfying Condition (8) of Definition 3.2 give rise to trivial examples of relative  $G$ -modular categories where  $G$  and  $\mathbf{Z}$  are both the trivial group,  $\overline{\mathbf{X}} = \emptyset$ ,  $\mathbf{A}$  is the set of simple objects and  $\mathbf{d} = \text{qdim}$  is the quantum dimension. In this case the construction proposed in this paper reduces to the original Reshetikhin-Turaev construction of [33, 34] and then  $\mathbf{N} = \mathbf{N}^0$  is the usual Reshetikhin-Turaev-Witten invariant.

Next we show how  $\mathbf{N}$  and  $\mathbf{N}^0$  behave under the connected sum defined in Equation (5). If  $T$  is a  $\mathbf{A}$ -graph in  $\mathbb{R}^3$  and  $T'$  is any  $\mathcal{C}$ -colored ribbon graph with coupons, then

$$(15) \quad F'(T \sqcup T') = F'(T)F(T').$$

Recall that the categorical dimension of an object  $V$  in  $\mathcal{C}$  is the element  $d'_V \circ b_V \in \text{End}(\mathbb{I}) = \mathbb{K}$ . There are many interesting examples where the categorical dimension is zero on all of the objects of  $\mathbf{A}$  (see Section 6 and the examples of [14, 15]). For such a category we have  $F(T) = 0$  for any

$A$ -graph and  $F'$  vanishes on a disjoint union of  $A$ -graphs (see Lemma 16 and Proposition 19 of [15]). Similarly, we have the following proposition:

**Proposition 3.14.** *Suppose that the categorical dimension of any object in  $A$  is zero. If  $(M, T, \omega)$  is  $T$ -admissible or has a computable surgery presentation then  $N^0(M, T, \omega) = 0$ . In addition, if  $(M', T', \omega')$  is a compatible triple, then*

$$N((M, T, \omega) \# (M', T', \omega')) = N(M, T, \omega) N^0(M', T', \omega').$$

*Proof.* We first prove the last equalities.

First if  $(M, T, \omega)$  is  $T$ -admissible then let  $(M, T_H, \omega_H)$  be a  $H$ -stabilization of  $M$  with computable presentation given by  $L \cup T_H$ . As in Definition 3.9, we call  $\alpha$  the color of the new meridian  $m \subset T_H$ . If  $\alpha$  has been chosen appropriately then  $(M', T', \omega') \# (S^3, u_\alpha, \omega_\alpha)$  as in Theorem 3.12 admits a computable presentation given by  $L' \cup T' \cup u_\alpha$ . Take a connected sum in  $S^3$  of these two presentations along  $m$  and  $u_\alpha$ . The property of  $F'$  given in Equation (13) implies that  $F'((L \cup T_H) \#_{(m, u_\alpha)} (L' \cup T' \cup u_\alpha)) = \frac{1}{d(\alpha)} F'(L \cup T_H) F'(L' \cup T' \cup u_\alpha)$ . But  $(L \cup T_H) \#_{(m, u_\alpha)} (L' \cup T' \cup u_\alpha)$  is a computable presentation of a  $H$ -stabilization of  $(M, T, \omega) \# (M', T', \omega')$ . Thus  $N((M, T, \omega) \# (M', T', \omega')) = N(M, T, \omega) N^0(M', T', \omega')$ .

Now if  $(M, T, \omega)$  has a computable surgery presentation, let  $L$  be a link which gives this surgery presentation. Let  $K$  be any sublink of  $L$  and let  $L_1 = L \setminus K$  then clearly  $F'(L \cup T) = F'(L_1 \cup (T \cup K))$ . If  $S_{L_1}^3$  is the manifold given by surgery on  $L_1$  then there exists a  $T$ -admissible triple  $(S_{L_1}^3, T \cup K, \omega)$  (where  $K$  is colored by the Kirby color  $\Omega_{g_\omega(K)}$ ) and  $a, b \in \mathbb{Z}$  such that

$$(16) \quad N(M, T, \omega) = \frac{N(S_{L_1}^3, T \cup K, \omega'')}{\Delta_+^b \Delta_-^a}.$$

This process can be thought of as taking the sublink  $K$  out of the surgery presentation and “adding” it to the graph  $T$ . To do this one changes the original manifold and so the signature changes; this is reflected in the constant mentioned above. Similarly, for the  $a, b \in \mathbb{Z}$  used in Equation (16) we have

$$(17) \quad N((M, T, \omega) \# (M', T', \omega')) = \frac{N((S_{L_1}^3, T \cup K, \omega'') \# (M', T', \omega'))}{\Delta_+^b \Delta_-^a}.$$

Now since  $(S_{L_1}^3, T \cup K, \omega'')$  is  $T$ -admissible then the argument in the previous paragraph shows that

$$N((S_{L_1}^3, T \cup K, \omega'') \# (M', T', \omega')) = N(S_{L_1}^3, T \cup K, \omega'') N^0(M', T', \omega').$$

The desired result follows from Equations (16) and (17).

Finally if  $(M, T, \omega)$  is  $T$ -admissible or has a computable surgery presentation, consider the previous equality with  $(M', T', \omega') = (S^3, u_\alpha, \omega_\alpha)$  the unknot in the 3-sphere colored by  $\alpha \in A$ . Then we have

$$N(M, T, \omega) N^0(S^3, u_\alpha, \omega_\alpha) = N((M, T, \omega) \# (S^3, u_\alpha, \omega_\alpha)) = N^0(M, T, \omega) N(S^3, u_\alpha, \omega_\alpha).$$

But  $N(S^3, u_\alpha, \omega_\alpha) = d(\alpha) \neq 0$  whereas  $N^0(S^3, u_\alpha, \omega_\alpha) = 0$  because  $F'$  vanishes on  $A$ -colored split links. Thus  $N^0(M, T, \omega) = 0$ . 3.14

**Proposition 3.15.** *If  $(M, T, \omega)$  and  $(M', T', \omega')$  are compatible triples, then*

$$N^0((M, T, \omega) \# (M', T', \omega')) = N^0(M, T, \omega) N^0(M', T', \omega')$$

*where the connected sum is taken over balls not intersecting  $T$  and  $T'$ .*



*Proof.* Let  $L \cup T \cup u_\alpha$  and  $L' \cup T' \cup u_\alpha$  be computable presentations respectively of  $(M, T, \omega) \# (S^3, u_\alpha, \omega_\alpha)$  and  $(M', T', \omega') \# (S^3, u_\alpha, \omega_\alpha)$ . Then since  $u_\alpha \# u_\alpha = u_\alpha$ , by the property of  $F'$  for connected sum it holds:

$$(18) \quad F'(L \cup L' \cup T \cup T' \cup u_\alpha) = d(\alpha)^{-1} F'(L \cup T \cup u_\alpha) F'(L' \cup T' \cup u_\alpha)$$

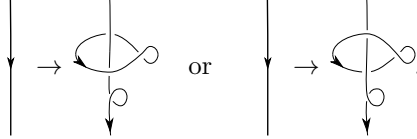
But  $L \cup L' \cup T \cup T' \cup u_\alpha$  provides a computable presentation for  $(M, T, \omega) \# (M', T', \omega') \# (S^3, u_\alpha, \omega_\alpha)$  and thus, using formula (18),  $N^0((M, T, \omega) \# (M', T', \omega'))$  can be computed from this presentation as

$$d(\alpha)^{-1} N((M, T, \omega) \# (M', T', \omega') \# (S^3, u_\alpha, \omega_\alpha)) = \frac{F'(L \cup T \cup u_\alpha) F'(L' \cup T' \cup u_\alpha)}{d(\alpha)^2 \Delta_+^{s+s'} \Delta_-^{p+p'}}$$

where  $s, p$  (resp.  $s', p'$ ) are the indices of inertia of the linking matrix of  $L$  and of  $L'$  respectively. But the latter is by definition  $N^0(M, T, \omega) N^0(M', T', \omega')$ . 3.15

#### 4. PROOFS OF THEOREMS 3.7 AND 3.10 AND PROPOSITION 3.8

**4.1. Idea of the proofs.** Recall that Kirby's theorem [24] allows one to relate any two presentations of an oriented 3-manifold as surgery over a framed link in  $S^3$  by means of handle-slides, blow-up and blow-down moves. *Handle-slides* are depicted schematically in the proof of Lemma 4.9: they consist in modifying one component of a link by replacing a chord of the component by one which is "slid" over or follows another component (and lies horizontal with respect to the framing). In this paper, *blow-up moves* consist in adding an unknot with framing  $\pm 1$  which is linked with one component of the link where the framing of this component is changed by  $\pm 1$ . A blow-up move with framing 1 or  $-1$  can be depicted by the local replacement:



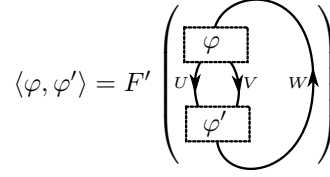
*Blow-down moves* are the inverse moves. The standard definition of a blow-up move is simply adding a  $\pm 1$ -framed unknot which is unlinked with the rest of the link; it is a standard fact that as soon as the link is not empty the standard definition is equivalent to our definition.

As remarked by R. Gompf and A. Stipsicz in [17] (see Theorem 5.3.6 and subsequent comments) every move  $L \rightarrow L'$  describes an isotopy class of diffeomorphisms between the surgered manifolds  $S_L^3$  and  $S_{L'}^3$ . So one can check that Kirby's theorem actually proves the following:

**Theorem 4.1** ([24]). *Let  $M_1$  and  $M_2$  be oriented 3-manifolds and  $f : M_1 \rightarrow M_2$  be an orientation preserving diffeomorphism. Any two surgery presentations  $L_1$  and  $L_2$  of  $M_1$  and  $M_2$ , respectively can be connected by a sequence of handle-slides, blow-up moves and blow-down moves such that the induced diffeomorphism between  $M_1 = S_{L_1}^3$  and  $M_2 = S_{L_2}^3$  is isotopic to  $f$ .*

The above theorem can be refined to the case of manifolds containing graphs (see [31]). In particular, we have the following:

**Theorem 4.2.** *Let  $M_1$  and  $M_2$  be oriented, closed 3-manifolds containing framed graphs  $T_1 \subset M_1$  and  $T_2 \subset M_2$ , respectively. Let  $f : M_1 \rightarrow M_2$  be an orientation preserving diffeomorphism such that  $f(T_1) = T_2$  as framed graphs. Let  $L_i$  be a link in  $S^3$  which is a surgery presentation of  $M_i$  such that  $T_i \subset S^3 \setminus L_i$ . There exists a sequence of handle-slides, blow-up moves and blow-down moves on the components of  $L_1$  as well as handle slides moving an edge of  $T_1$  over a component of  $L_1$  and blow-up and blow-down moves around edges of  $T_1$ , such that the induced diffeomorphism between  $M_1 = S_{L_1}^3$  and  $M_2 = S_{L_2}^3$  is isotopic to  $f$ .*

FIGURE 4. Pairing of morphisms in  $\mathcal{C}$ .

We will now discuss the idea of the proofs of Theorem 3.7 and 3.10. Suppose  $(M, T, \omega)$  and  $(M', T', \omega')$  are two compatible triples which are diffeomorphic through a map  $f : (M, T, \omega) \rightarrow (M', T', \omega')$  (i.e.  $f(T) = T'$  as framed graph and  $f^*(\omega') = \omega$ ). Pick any two computable surgery presentations of  $(M, T, \omega)$  and  $(M', T', \omega')$  through links  $L$  and  $L'$  in  $S^3$ . Use Theorem 4.1 to realize the map  $f$  through a sequence of handle-slides, blow-up moves and blow-down moves; then try to follow the sequence and prove that the values of the invariants before and after each move does not change. In order to do so, we have to deal not only with framed links in  $S^3$  but also with their colorings and how they change during the moves. We will show in Lemma 4.9 that when a component of a link slides over a second one, the color of the latter is modified. A similar phenomenon happens when a blow-up/down move is applied (see Lemma 4.10). But our invariants are defined only when the colors of all the components of the links are “generic” (for instance in the case of  $\mathfrak{sl}_2$  when they belong to  $\mathbb{C} \setminus \mathbb{Z}$ ); so, if during a sequence of moves a non-generic color is produced the corresponding invariant is not defined and the invariance cannot be proven directly. To bypass this problem we initially apply an  $H$ -stabilization along an edge  $e$  of  $T$  which is colored by an element of  $A$  and color the newly created meridian of  $e$  by a “sufficiently generic” color. Then whenever a move in the sequence would produce a non-generic color, before doing the move we slide the meridian over it to change its color to a generic one. Then, at the end of the sequence we slide back the meridian in its “original position” around  $e$  and remove it.

In the proof of Theorem 3.7 we do not have an edge  $e$  of  $T$  to create such a useful meridian, but we may apply a blow-up move over a component of the surgery link. This produces a meridian colored by a linear combination of generic colors. We then consider this meridian as part of the graph  $T$  and our invariant as a linear combination of invariants where  $T$  is non-empty and has a generically colored component; we thus may apply the preceding argument.

**4.2. Useful lemmas.** Before proving Theorems 3.7 and 3.10 we prove a series of lemma which will be used in the proof. If  $U, V, W$  are objects of  $\mathcal{C}$ , the duality morphisms of  $\mathcal{C}$  induce natural isomorphisms

$$(19) \quad \begin{aligned} \mathrm{Hom}_{\mathcal{C}}(U \otimes V, W) &\cong \mathrm{Hom}_{\mathcal{C}}(U, W \otimes V^*) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, W \otimes V^* \otimes U^*) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V^* \otimes U^* \otimes W). \end{aligned}$$

Furthermore, the value of  $F'$  on the planar theta-graph with coupons and edges colored by  $U, V$  and  $W$  (see Figure 4) gives a pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}_{\mathcal{C}}(U \otimes V, W) \otimes \mathrm{Hom}_{\mathcal{C}}(W, U \otimes V) \rightarrow \mathbb{K}$$

which is compatible with these isomorphisms.

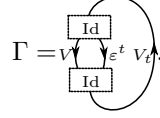
By definition of  $F'$  and this pairing, we have

$$\varphi \circ \varphi' = \mathrm{d}(W)^{-1} \langle \varphi, \varphi' \rangle \mathrm{Id}_W$$

for any  $\varphi \in \mathrm{Hom}_{\mathcal{C}}(U \otimes V, W)$  and  $\varphi' \in \mathrm{Hom}_{\mathcal{C}}(W, U \otimes V)$ .

**Lemma 4.3.** *Let  $V \in \mathbf{A}$  and  $t \in \mathbf{Z}$  be such that  $V \otimes \varepsilon^t \in \mathbf{A}$ . Then  $d(V \otimes \varepsilon^t) = d(V)$ . In other words,  $d$  factors through a map on the simple  $\mathbf{Z}$ -orbits.*

*Proof.* Let  $V_t = V \otimes \varepsilon^t$  and consider the following  $\mathcal{C}$ -colored ribbon graph with coupons:



To compute  $F'(\Gamma)$ , one can cut the edge colored by  $V_t$ . The image under  $F$  of the obtained tangle is  $\text{Id}_{V_t}$  so  $F'(\Gamma) = d(V_t)$ . On the other hand, one can cut the edge colored by  $V$  and so

$$F'(\Gamma) \text{Id}_V = F \left( \text{tangle with } V \text{ and } V_t \right) = d(V) F \left( \text{tangle with } V \text{ and } \varepsilon^t \right) = d(V) \text{qdim}(\varepsilon^t) \text{Id}_V = d(V) \text{Id}_V.$$

Hence  $d(V_t) = F'(\Gamma) = d(V)$ . 4.3

The following proposition gives a relationship between vanishings of  $F$  and  $F'$ .

**Proposition 4.4.** *Let  $\bar{V} = ((V_1, \varepsilon_1), \dots, (V_k, \varepsilon_k))$  and  $\bar{W} = ((W_1, \varepsilon'_1), \dots, (W_l, \varepsilon'_l))$  be objects of  $\text{Rib}_{\mathcal{C}}$  such that at least one of the  $V_i$  belongs to  $\mathbf{A}$  and for any  $1 \leq i \leq k$  and  $1 \leq j \leq l$  the objects  $V_i$  and  $W_j$  are in different graded pieces of  $\mathcal{C}$ . Suppose that  $T$  is a  $\mathcal{C}$ -colored ribbon graph in  $\text{Hom}_{\text{Rib}_{\mathcal{C}}}(\bar{V}, \bar{W})$ . Then the following two are equivalent*

- $F(T) = 0$
- For all  $T' \in \text{Hom}_{\text{Rib}_{\mathcal{C}}}(\bar{W}, \bar{V})$ , one has  $F'(\text{tr}(T \circ T')) = 0$  where  $\text{tr}(T \circ T')$  is the trace in  $\text{Rib}_{\mathcal{C}}$ , i.e. the braid closure of the tangle formed from  $T'$  on top of  $T$ .

*Proof.* Assume that  $F(T) = 0$  and that  $T'$  is as in the proposition. Then one can compute  $F'(\text{tr}(T \circ T'))$  by cutting the edge of  $T$  colored by  $V_i \in \mathbf{A}$ . This produce a 1-1-tangle containing  $T$  as a subgraph and thus the functor  $F$  vanish on it. Hence  $F'(\text{tr}(T \circ T')) = 0$ .

Suppose now that  $F(T) = f \neq 0$ . Let  $g \in G$  be such that  $F(\bar{V}) = V_1 \otimes \dots \otimes V_k \in \mathcal{C}_g$ . As  $f \neq 0$ , the  $G$ -grading impose that  $F(\bar{W}) = W_1 \otimes \dots \otimes W_l \in \mathcal{C}_g$ . Let  $h \in G \setminus (X_r \cup g^{-1}X_r)$  and let  $U$  be a simple object of  $\mathcal{C}_h$ . Then  $\bar{V} \otimes U$  and  $\bar{W} \otimes U$  are semi-simple objects because they belongs to  $\mathcal{C}_{gh}$  where  $gh \notin X_r$ . Now  $f \otimes \text{Id}_U \neq 0$  so there is a simple object  $U' \in \mathcal{C}_{gh}$  and maps  $i : U' \rightarrow \bar{V} \otimes U$ ,  $p : \bar{W} \otimes U \rightarrow U'$  such that  $p \circ (f \otimes \text{Id}_U) \circ i = \text{Id}_{U'}$ . Let  $T'' \supset T$  be the ribbon graph with coupons colored by  $p$  and  $i$  be the graph corresponding to the expression  $p \circ (f \otimes \text{Id}_U) \circ i$ . Then  $F(T'' - \text{Id}_{(U', +)}) = 0$  hence the braid closures satisfy  $F'(\text{tr}(T'')) = F'(\text{tr}(\text{Id}_{(U', +)})) = d(U')$ . Now for  $T'$  the complement of  $T$  in  $\text{tr}(T'')$ , we have  $F'(\text{tr}(T \circ T')) = d(U') \neq 0$ . 4.4

**Lemma 4.5** (Fusion lemma). *Let  $g \in G \setminus \bar{X}$ . Let  $U, V \in \mathcal{C}$ , with  $U \otimes V \in \mathcal{C}_g$  and let  $W$  be a simple object of  $\mathcal{C}_g$ . Then the pairing  $\langle \cdot, \cdot \rangle : \text{Hom}_{\mathcal{C}}(U \otimes V, W) \otimes \text{Hom}_{\mathcal{C}}(W, U \otimes V) \rightarrow \mathbb{K}$  is non degenerate. Furthermore,*

$$(20) \quad \text{Id}_{U \otimes V} = \sum_W \sum_i d(W) x^{W,i} \circ x_{W,i}$$

where  $W$  runs through a representative set of isomorphism classes of simple objects of  $\mathcal{C}_g$ ,  $(x_{W,i})_i$  is a basis of  $\text{Hom}_{\mathcal{C}}(U \otimes V, W)$  and  $(x^{W,i})_i$  is the dual basis of  $\text{Hom}_{\mathcal{C}}(W, U \otimes V)$ . Let  $(W_j)_{j=1 \dots n}$  be a finite set of simple modules of  $\mathcal{C}_g$  whose isomorphism classes represent all simple  $\mathbf{Z}$ -orbits

of  $\mathcal{C}_g$ . Then the above sum can be rewritten as

$$\text{Id}_{U \otimes V} = \sum_{j=1}^n \mathbf{d}(W_j) \sum_{t \in \mathbb{Z}} \sum_i x^{j,t,i} \circ x_{j,t,i}$$

where  $(x_{j,t,i})_i$  is a basis of  $\text{Hom}_{\mathcal{C}}(U \otimes V, W_j \otimes \varepsilon^t)$  and  $(x^{j,t,i})_i$  is the dual basis of  $\text{Hom}_{\mathcal{C}}(W_j \otimes \varepsilon^t, U \otimes V)$ .

*Proof.* As  $\mathcal{C}_g$  is semi-simple, we can write  $U \otimes V \simeq \bigoplus_k W_k^{\oplus n_k}$  where  $W_k$  are non-isomorphic simple objects of  $\mathcal{C}_g$ . So there are maps  $x_{W_k,i} : U \otimes V \rightarrow W_k$  and  $y^{W_k,i} : W_k \rightarrow U \otimes V$  such that  $x_{W_k,i} \circ y^{W_k,j} = \delta_i^j \text{Id}_{W_k}$  and  $\text{Id}_{U \otimes V} = \sum_k \sum_i y^{W_k,i} \circ x_{W_k,i}$ . Now if  $\varphi \in \text{Hom}_{\mathcal{C}}(U \otimes V, W)$  is not zero, then  $\varphi = \sum_k \sum_i \varphi \circ y^{W_k,i} \circ x_{W_k,i}$  so there exist  $k, i$  with  $\varphi \circ y^{W_k,i} \neq 0 \in \text{Hom}_{\mathcal{C}}(W_k, W)$ . This implies that  $W_k$  and  $W$  are isomorphic and if  $\varphi' = y^{W_k,i} \circ (\varphi \circ y^{W_k,i})^{-1} \in \text{Hom}_{\mathcal{C}}(W, U \otimes V)$ , then we have  $\varphi \circ \varphi' = \text{Id}_W$  implying  $\langle \varphi, \varphi' \rangle = \mathbf{d}(W) \neq 0$ . Hence the pairing is not degenerate.

Futhermore,  $x_{W_k,i} \circ y^{W_k,j} = \delta_i^j \text{Id}_{W_k}$  so  $\langle x_{W_k,i}, y^{W_k,j} \rangle = \delta_i^j \mathbf{d}(W_k)$  which implies that the basis  $(\mathbf{d}(W_k)^{-1} y^{W_k,i})_{i=1 \dots n_k}$  is the dual basis of  $(x_{W_k,i})_{i=1 \dots n_k}$ . Hence the identity of  $U \otimes V$  can be decomposed as in Equation (20). Remark that for all but a finite number of simple modules  $W$  the corresponding term in the sum is zero so that the sum in Equation (20) is finite.

We can then rewrite the sum by grouping the simple objects  $W$  that are in the same simple  $\mathbb{Z}$ -orbit. Here we use that the action of  $\mathbb{Z}$  is free so that any  $W$  is isomorphic to a unique module of the form  $W_j \otimes \varepsilon^t$ . Then using that  $\mathbf{d}(W_j \otimes \varepsilon^t) = \mathbf{d}(W_j)$  we obtain the second decomposition of  $\text{Id}_{U \otimes V}$ . 4.5

Let  $T$  be a homogeneous  $\mathcal{C}$ -colored ribbon graph with coloring  $\varphi$ , whose associated  $G$ -coloring we denote  $\bar{\varphi}$ . Let  $K$  be an oriented knot in the complement of  $T$  and let  $\Sigma$  be a Seifert surface for  $K$ . Then the homological intersection of the homology class relative to  $K$  represented by  $\Sigma$  with  $\bar{\varphi}$  is an element of  $G$  depending on  $K$ . Let

$$\text{lk}^G(K, \bar{\varphi}) = \prod_{p \in \Sigma \cap T} \bar{\varphi}(e_p)^{\text{sign}(p)}$$

where  $e_p$  is the edge of  $T$  containing  $p$  and  $\text{sign}(p)$  is the sign of the intersection of  $\Sigma$  and  $e_p$  at  $p$ . Clearly  $\text{lk}^G(K, \bar{\varphi})$  depends only on  $[K] \in H_1(S^3 \setminus T; \mathbb{Z})$  and it can be computed from a diagram of  $T$  by the following rules: (1) if  $e$  is an edge of  $T$  which is below  $K$  then  $e$  does not contribute to  $\text{lk}^G(K, \bar{\varphi})$ , (2) if  $e$  is above  $K$  and the pair  $(K, e)$  form a positive crossing then  $e$  contributes  $\bar{\varphi}(e)$  to  $\text{lk}^G(K, \bar{\varphi})$ , (3) if  $e$  is above  $K$  and the pair  $(K, e)$  form a negative crossing then  $e$  contributes  $\bar{\varphi}(e)^{-1}$  to  $\text{lk}^G(K, \bar{\varphi})$ . These rules can be summarized by:

$$(21) \quad \begin{array}{c} K \quad e \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{---} \end{array} \rightarrow \bar{\varphi}(e), \quad \begin{array}{c} e \quad K \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{---} \end{array} \rightarrow \bar{\varphi}(e)^{-1}, \quad \begin{array}{c} K \quad e \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{---} \end{array} \rightarrow 1, \quad \begin{array}{c} e \quad K \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{---} \end{array} \rightarrow 1,$$

where here we are using a multiplicative notation for the group  $G$ . We extend the above definition to the case when  $K \subset T$  by defining  $\text{lk}^G(K, \bar{\varphi})$  to be  $\text{lk}^G(K_{\parallel}, \bar{\varphi})$  where  $K_{\parallel}$  is a parallel copy of  $K$  given by the framing of  $T$ . In particular, if  $L \cup T$  is a  $\mathcal{C}$ -colored ribbon graph with  $G$ -coloring  $\bar{\varphi}$  and linking matrix  $\text{lk}$ , and if  $K_i$  is the  $i^{\text{th}}$  component of  $L$  then

$$\text{lk}^G(K_i, \bar{\varphi}) = \prod_j \bar{\varphi}(K_j)^{\text{lk}_{ij}} = \prod_j \bar{\varphi}(K_j)^{\text{lk}_{ji}}$$

even if  $\text{lk}$  is not symmetric. Thus, if  $\bar{\varphi}$  is in the image of  $\Phi$  then  $\text{lk}^G(K_i, \bar{\varphi}) = 1 \in G$ . In particular, we have the following remark.

**Remark 4.6.** Let  $(M, T, \omega)$  be a compatible triple with a computable surgery presentation via  $L$ . Let  $g_\omega$  be the  $G$ -coloring on  $L \cup T$  coming from  $\omega$ . If  $L_i$  is any component of  $L$  then  $\text{lk}^G(L_i, g_\omega) = 1$ .

**Lemma 4.7.** *Let  $T$  be a  $\mathbf{A}$ -graph with corresponding  $G$ -coloring  $\bar{\varphi}$ . Suppose  $K$  is a circle component of  $T$  colored by  $\varepsilon^t$  for some  $t \in \mathbb{Z}$ . Let  $T'$  be the  $\mathbf{A}$ -graph  $T$  with the component  $K$  removed. Then*

$$F'(T') = \text{lk}^G(K, \bar{\varphi})^{\bullet t} F'(T).$$

*Proof.* We consider a diagram representing  $T$  and use the skein relation (12) to pull the knot  $K$  above the rest of the diagram, obtaining the disjoint union of  $T' = T \setminus K$  with  $K$ . Using Equation (21), one can see that  $F'(T) = \text{lk}^G(K, \bar{\varphi})^{\bullet t} F'((T \setminus K) \sqcup K)$ . Combining this equality with Equation (15) we have

$$F'(T) = \text{lk}^G(K, \bar{\varphi})^{\bullet t} F'(T' \sqcup K) = \text{lk}^G(K, \bar{\varphi})^{\bullet t} F'(T') F(K)$$

and the lemma follows from the fact that  $F(K) = 1$  which can be easily deduced from the braiding and modified dimension of  $\varepsilon^t$  since  $c_{\varepsilon^t, \varepsilon^t} \circ c_{\varepsilon^t, \varepsilon^t} = \text{Id}_{\varepsilon^t \otimes \varepsilon^t}$  and  $\text{qdim}(\varepsilon^t) = 1$ . 4.7

**Lemma 4.8.** *Let  $T$  be a  $\mathbf{A}$ -graph with  $G$ -coloring  $\bar{\varphi}$ . Suppose  $K$  is a circle component of  $T$  colored by  $V \in \mathbf{A}$  with  $\text{lk}^G(K, \bar{\varphi}) = 1$ . Let  $T'$  be the  $\mathbf{A}$ -graph  $T$  where the color of the component  $K$  is changed to  $V \otimes \varepsilon^t$  for some  $t \in \mathbb{Z}$  then  $F'(T') = F'(T)$ .*

*Proof.* Let  $T'' = T \cup K_{\parallel}$  where  $K_{\parallel}$  is a parallel copy of  $K$  colored with  $\varepsilon^t$ . Then basic properties of  $F$  imply  $F'(T') = F'(T'')$ . The  $G$ -coloring  $\varphi$  of  $T$  extends to a  $G$ -coloring  $\bar{\varphi}'$  of  $T''$  with value 1 on  $K_{\parallel}$ . Now one can apply Lemma 4.7 to  $T''$  which gives  $F'(T'') = \text{lk}^G(K_{\parallel}, \bar{\varphi}')^{\bullet t} F'(T)$ . But  $\text{lk}^G(K_{\parallel}, \bar{\varphi}') = \text{lk}^G(K, \bar{\varphi}) = 1$  so  $F'(T') = F'(T)$ . 4.8

**Lemma 4.9** (Handle slide). *Let  $T$  be a  $\mathcal{C}$ -colored ribbon graph with a homogeneous  $\mathcal{C}$ -coloring  $\varphi$ . Suppose  $K$  is a circle component of  $T$  colored by a Kirby color  $\Omega_g$  of degree  $g \in G \setminus \bar{X}$ . Let  $e$  be an oriented edge of  $T \setminus K$  homogeneously colored by  $\varphi(e) \in \mathcal{C}$ . Let  $T'$  be a  $\mathcal{C}$ -colored ribbon graph obtained from  $T$  by a handle-slide of  $e$  along  $K$  with the color  $\Omega_g$  of  $K$  replaced by a Kirby color  $\Omega_h$  of degree  $h = \bar{\varphi}(e)g$ . If  $\text{lk}^G(K, \bar{\varphi}) = 1$  and  $h \notin \bar{X}$  then  $F'(T') = F'(T)$ .*

*Proof.* Let  $O(g)$  and  $O(h)$  be the sets of colors that appear in  $\Omega_g$  and  $\Omega_h$ , respectively. Since  $\text{lk}^G(K, \bar{\varphi}) = 1$ , Lemma 4.8 implies that  $T$  can be modified by adding a parallel copy of  $K$  colored by  $\varepsilon^t$  without changing its value under  $F'$ . We use this fact with the fusion of Lemma 4.5 to prove the lemma. In particular, let  $U$  be one of the modules appearing in  $\varphi(e)$ . We will show that the edge  $e$  colored by  $U$  can slide over  $K$ , then the result follows since  $\varphi(e)$  is a the formal linear combination of modules. In the following equation the symbol  $\stackrel{\bullet}{=}$  displays the equality of

the values of the corresponding local diagrams under  $F'$ .

$$\begin{aligned}
\sum_{V \in O(g)} d(V) \left( \begin{array}{c} U \\ \downarrow \\ V \end{array} \right) &\stackrel{\bullet}{=} \sum_{\substack{(V, W) \in O(g) \times O(h) \\ t \in \mathbb{Z}, W_t = W \otimes \varepsilon^t}} d(V) d(W) \sum_i \left( \begin{array}{c} U \quad V \\ \downarrow \quad \downarrow \\ x_i \quad x^i \\ \downarrow \quad \downarrow \\ U \quad V \end{array} \right) \\
&\stackrel{\bullet}{=} \sum_{\substack{(V, W) \in O(g) \times O(h) \\ t \in \mathbb{Z}, W_t = W \otimes \varepsilon^t}} d(V) d(W) \sum_i \left( \begin{array}{c} U \quad V \\ \downarrow \quad \downarrow \\ x_i \quad x^i \\ \downarrow \quad \downarrow \\ U \quad V \end{array} \right) \varepsilon^{-t} \\
&\stackrel{\bullet}{=} \sum_{\substack{(V, W) \in O(g) \times O(h) \\ t \in \mathbb{Z}, V_t = V \otimes \varepsilon^{-t}}} d(V) d(W) \sum_i \left( \begin{array}{c} U \quad V_t \\ \downarrow \quad \downarrow \\ y_i \quad y^i \\ \downarrow \quad \downarrow \\ U \quad V \end{array} \right) \\
&\stackrel{\bullet}{=} \sum_{\substack{(V, W) \in O(g) \times O(h) \\ t \in \mathbb{Z}, V_t = V \otimes \varepsilon^{-t}}} d(V) d(W) \sum_i \left( \begin{array}{c} U \quad W \\ \downarrow \quad \downarrow \\ z_i \quad z^i \\ \downarrow \quad \downarrow \\ U \quad V_t \end{array} \right) \\
&\stackrel{\bullet}{=} \sum_{W \in O(h)} d(W) \left( \begin{array}{c} U \\ \downarrow \\ W \end{array} \right)
\end{aligned}$$

In these sums,  $\{x_i\}_i$  and  $\{x^i\}_i$  are arbitrary dual bases of the multiplicity modules  $\text{Hom}(W_t, U \otimes V)$  and  $\text{Hom}(U \otimes V, W_t)$  respectively;  $\{y_i\}_i$  is a base deduced from  $\{x_i\}_i$  using Lemma 3.1,  $\{y^i\}_i$  is its dual bases;  $\{z_i\}_i$  is a basis obtained from  $\{y^i\}_i$  using the isomorphisms of the multiplicity modules and  $\{z^i\}_i$  denotes its dual bases. The first and last equality are obtained from Lemma 4.5, the second from Lemma 4.7, the third comes from Lemma 3.1 and the fourth is an isotopy plus a local modification of the coupons. 4.9

**Lemma 4.10.** *There exists scalars  $\Delta_{\pm} \in \mathbb{K}$  such that for any module  $V$  of degree  $g \in G \setminus \overline{X}$  we have*

$$(22) \quad F \left( \begin{array}{c} \text{loop} \\ \downarrow \\ V \end{array} \right) = \Delta_- \text{Id}_V \quad \text{and} \quad F \left( \begin{array}{c} \text{loop} \\ \downarrow \\ V \end{array} \right) = \Delta_+ \text{Id}_V.$$

*Proof.* Suppose first  $V \in \mathbf{A}$  and let  $\Delta_{\pm}(V) \in \mathbb{K}$  be the quantities determined by Equation (22). We want to show that these quantities do not depend on  $V$ . We will do this for  $\Delta_-$ ; a similar argument implies the result for  $\Delta_+$ . If  $g' \in G \setminus \overline{X}$  then there exists  $h \in G \setminus (\overline{X} \cup (g^{-1}\overline{X}) \cup (g'^{-1}\overline{X}))$ . Let  $W$  be a simple object of  $\mathcal{C}_h$ . Now for any  $V \in \mathbf{A}$  of degree  $g$ , we can use the handle slide

property of Lemma 4.9 and Proposition 4.4 to obtain

$$F \left( \begin{array}{c} \text{Diagram 1} \end{array} \right) = F \left( \begin{array}{c} \text{Diagram 2} \end{array} \right) = F \left( \begin{array}{c} \text{Diagram 3} \end{array} \right).$$

The left (resp. right) hand side of the last equation is equal to  $\Delta_-(V) \text{Id}_{V \otimes W}$  (resp.  $\Delta_-(W) \text{Id}_{V \otimes W}$ ) so and  $\Delta_-(V) = \Delta_-(W)$ . Similarly, for any  $V' \in \mathbf{A}$  of degree  $g'$  we have  $\Delta_-(V') = \Delta_-(W)$  and so  $\Delta_-(V) = \Delta_-(V')$ .

Now let  $V$  be any object of the semi-simple category  $\mathcal{C}_g$  and let  $f_V \in \text{End}_{\mathcal{C}}(V)$  be the endomorphism represented by the first ribbon graph in the lemma. Then  $V = \bigoplus_i V_i$  with  $V_i \in \mathbf{A}$ . Let  $\alpha_i : V_i \rightarrow V$  and  $\beta_i : V \rightarrow V_i$  for  $i = 1, \dots, n$  such that  $\text{Id}_V = \sum_{i=1}^n \alpha_i \beta_i$ . Then  $f_V = \sum_{i=1}^n f_V \alpha_i \beta_i = \sum_{i=1}^n \alpha_i \Delta_-(V_i) \beta_i = \Delta_- \text{Id}_V$ . 4.10

**Lemma 4.11.** *Let  $(M, T, \omega)$  be a compatible triple and consider  $\omega$  as a map on  $H_1(M \setminus T, \mathbb{Z})$  with values in  $G$ . Suppose that  $g \in G$  is a value of  $\omega$ , then there exists a surgery presentation  $L \cup T$  of  $(M, T, \omega)$  for which the  $G$ -color of a component of  $L$  is  $g$ .*

*Proof.* Start with any presentation  $L' \cup T$  of  $(M, T, \omega)$ .  $H_1(M \setminus T, \mathbb{Z})$  is generated by the meridians around the edges of  $L' \cup T$ . So there exists integers  $n_i \in \mathbb{Z}$  such that  $\omega(\sum_i n_i m_i) = \prod_i g_{\omega}(e_i)^{n_i} = g$  where  $\{m_i\}_i$  is the set of meridians around the edges  $\{e_i\}_i$  of  $L' \cup T$ . We add to  $L'$  a disjoint unknot  $U$  with framing 1. The  $G$ -color of  $U$  is  $1 \in G$ . Then we can slide  $n_i$  times the edge  $e_i$  on  $U$  so that the resulting  $G$ -coloring of  $U$  becomes  $g_{\omega}(U) = g$ . For  $L = L' \cup U$ , we get the desired presentation of  $(M, T, \omega)$ . 4.11

### 4.3. Proofs.

*Proof of Theorem 3.10.* Since  $(M, T, \omega)$  is  $T$ -admissible then  $T \neq \emptyset$  and has an edge  $e$  colored by  $\beta \in \mathbf{A}$ . Pick a surgery presentation of  $(M, T, \omega)$  over a link  $L \subset S^3$ . If  $L = \emptyset$  then since  $e \in T$  is colored by  $\beta \in \mathbf{A}$  we have  $L$  is computable. Suppose now that the presentation is not computable. Then  $L \neq \emptyset$  and let  $L_i$  be the components of  $L$  where  $i = 1, \dots, n$ . Then let  $S = \{i \in \{1, \dots, n\} : g_{\omega}(L_i) \in \overline{X}\}$ . Apply a  $H$ -stabilization along  $e$  to create a new component  $m$  colored by  $\alpha \in \mathbf{A}$  where  $\alpha$  is chosen so that  $\bar{\alpha} g_{\omega}(L_i) \notin \overline{X}$  for all  $i \in S$ . Then one can do handle-slides to slide  $m$  over all the components  $L_i$  with  $i \in S$ . The resulting presentation is computable and this proves the first statement.

To prove the second statement, we first let  $L$  and  $L'$  be two computable surgery presentations of  $(M, T', \omega')$  where  $(M, T', \omega')$  is obtained by  $H$ -stabilization over the same edge  $e \subset T$  colored by  $\beta \in \mathbf{A}$ . In both stabilizations, let  $m$  be the component added to  $T$  and let  $\alpha \in \mathbf{A}$  be the color of  $m$ .

Let  $L = L^0 \xrightarrow{s_1} \dots \xrightarrow{s_k} L^k = L'$  be a sequence of handle-slides, blow-up moves and blow-down moves connecting the two presentations and inducing a diffeomorphism  $f$  between  $S_L^3 \setminus T'$  and  $S_{L'}^3 \setminus T'$  such that  $f^*(\omega') = \omega$ . The  $H$ -stabilization adds a meridian  $m$  of  $e$  to  $T$  colored by  $\alpha \in \mathbf{A}$ . We choose  $\alpha$  “generically” with respect to the sequence  $L^0 \xrightarrow{s_1} \dots \xrightarrow{s_k} L^k$  where generically means that  $\alpha$  is chosen by the finite list of conditions given below.

Suppose that  $s_1$  is a handle-slide of an edge of  $L^0 \cup T$  colored by  $\gamma$  over a component  $L_i^0$  of  $L^0$  colored by  $g_{\omega}(L_i^0) \in G$  (recall that  $g_{\omega}(L_i^0) \in G$  was defined as the value of  $\omega$  on the meridian of  $L_i^0$ ; see Definition 3.5). If  $g_{\omega}(L_i^0) \bar{\gamma} \in G \setminus \overline{X}$  then  $L^1$  is a computable presentation and we can apply Lemma 4.9. Otherwise, suppose that  $\alpha$  is such that  $\bar{\alpha} g_{\omega}(L_i^0), \bar{\alpha} g_{\omega}(L_i^0) \bar{\gamma} \in G \setminus \overline{X}$ ; this imposes two conditions on  $\alpha$  which are part of our genericity hypothesis. Slide  $m$  over  $L_i^0$  and perform the handle slide  $s_1$ . The result is  $L^1$  with a new component  $m$  colored by  $\alpha$  and where

the color of  $L_i^1$  changed to  $g_{\omega_1}(L_i^1)\bar{\alpha}$ ; the rest of  $L^1$  is unchanged. On the other hand, if  $s_1$  is a blow-up move then we can slide  $m$  over the newly-created component and color the latter by  $\bar{\alpha}$ . In both cases the resulting presentation will be computable and will contain a component  $m$  colored by  $\alpha$ .

Proceed as above to follow the rest of the sequence: each time we need to slide  $m$  over a component to make a handle-slide computable we add some conditions on  $\alpha$  to our list. The list of conditions imposed will be finite and by Hypothesis (4) of Definition 3.2 there exists an  $\alpha$  such that all of them are fulfilled.

At the last step one gets computable presentation via a link  $L'$  of the compatible triple  $(M, T', \omega')$  where  $T'$  is  $T$  together with a new meridian  $m$  colored by  $\alpha$ . Notice that no slides on  $m$  have occurred. Therefore, since  $m$  is by construction isotopic to a meridian of  $e$  in  $M$  one can find an isotopy of  $m$  in  $M$  which brings it in the position of a small meridian around  $e$  in the presentation  $L'$  where  $L'$  is colored by  $g_{\omega'}$ . This isotopy can be decomposed into a sequence  $L' \cup m \xrightarrow{h_1} \dots \xrightarrow{h_l} L' \cup m$  of handle slides of  $m$  over the components of  $L'$ . At each step  $s \in \{1, \dots, l\}$  the color of a component  $L'_i$  of  $L'$  will be of the form  $g_{\omega'}(L'_i)\bar{\alpha}^{x_s^i}$  for some  $x_s^i \in \mathbb{Z}$  depending on both the component  $L'_i$  and the step  $s$ . At the end of the isotopy the color of the component  $L'_i$  of  $L'$  is  $g_{\omega'}(L'_i)$  and so  $x_l^i = 0$  for all  $i$ . Thus in order to be able to perform the sequence of slidings  $h_s$  of  $m'$  over the components of  $L'$  bringing  $m'$  back to its position of meridian of  $e$  it is sufficient to add to the above list of genericity conditions the conditions

$$g_{\omega'}(L'_i)\bar{\alpha}^{x_s^i} \notin \bar{X}$$

for every component  $L'_i$  of  $L$  and for all  $s \in \{1, \dots, l-1\}$ . The union of these conditions and those found precedingly can be satisfied for a suitable choice of  $\alpha$  since no union of finitely many translates of  $\bar{X}$  covers  $G$  (see Definition 3.2).

Thus, we obtain a sequence of moves connecting the two presentations provided by  $L$  and  $L'$  (with one  $\alpha$ -colored meridian  $m$  added around  $e$ ) of the  $H$ -stabilization  $(M, T', \omega')$  through computable presentations. Thus applying Lemma 4.9 and Lemma 4.10 one has

$$(23) \quad \frac{F'(L \cup T \cup m)}{\Delta_+^r \Delta_-^s} = \frac{F'(L' \cup T \cup m)}{\Delta_+^{r'} \Delta_-^{s'}}.$$

where  $r, s$  (resp.  $r', s'$ ) the number of positive and negative eigenvalues of the linking form of  $L$  (resp.  $L'$ ). Now observe that  $F'(L \cup T \cup m) = F'(L \cup T)\langle H \rangle$  where  $\langle H \rangle$  is the value of the long Hopf-link whose closed component is colored by  $\alpha$  and whose long component is colored by  $\beta$ . Similarly  $F'(L' \cup T \cup m) = F'(L' \cup T)\langle H \rangle$ . Definition 3.2 implies  $\langle H \rangle \neq 0$  and so

$$\frac{F'(L \cup T)}{\langle H \rangle \Delta_+^r \Delta_-^s} = \frac{F'(L' \cup T)}{\langle H \rangle \Delta_+^{r'} \Delta_-^{s'}}.$$

This proves that invariant is well defined for surgery presentations of  $H$ -stabilization over the edge  $e$ . To conclude the proof it is sufficient to show that we can choose an  $H$ -stabilization over another edge  $e'$  of  $T$ . To see this one can apply a  $H$ -stabilization to both  $e$  and  $e'$  to easily see that the values of the two invariants are the same. 3.10

*Proof of Proposition 3.8.* By Lemma 4.11, there exists a surgery presentation  $L \cup T$  of  $(M, T, \omega)$  with an edge  $e$  such that for each  $x \in \bar{X}$  there exists  $n(x) \in \mathbb{Z}$  such that  $xg_{\omega}(e)^{n(x)} \notin \bar{X}$ . If  $L_i$  is a component of  $L$  such that  $g_{\omega}(L_i) \in \bar{X}$  then sliding  $e$  over  $L_i$   $n(g_{\omega}(L_i))$ -times the color of  $L_i$  is changed to a color not in  $\bar{X}$ . Thus, by sliding  $e$  when needed we obtain a computable presentation. 3.8



*Proof of Theorem 3.7.* If  $L$  and  $L'$  are two computable presentations of  $(M, T, \omega)$  then there exists a sequence of handle-slides, blow-up moves and blow-down moves  $L = L^0 \xrightarrow{s_1} \dots \xrightarrow{s_k} L^k = L'$  connecting them and inducing a diffeomorphism  $f$  such that  $f^*(\omega) = \omega$ . Pick a component  $L_0^0$  of  $L^0$  colored by  $g_0$  and apply a blow-up move to create a meridian  $m$  colored by  $g_0 \in G \setminus \bar{X}$  (see Lemma 4.10). Without loss of generality one can suppose that, if  $L_0^0$  is destroyed by a blow-down move during the sequence, then this happens at the very last step  $s_k$ .

Follow the sequence “ignoring  $m$ ”: when a component  $L_i$  slides over  $L_0^0$  it gets linked (or unlinked) with  $m$ . At the end of the sequence we have a link  $L' \cup m$  where the components of  $L'$  are colored by  $g_\omega(L'_i)$  and  $m$  is colored by  $g_0$ . The Kirby color  $\Omega_{g_0}$  of  $m$  is a formal finite linear combination of typical colors:  $\Omega_{g_0} = \sum_\alpha d(V_\alpha) V_\alpha$ , where  $\alpha \in \mathbf{A}$  because  $g_0 \notin \bar{X}$ . Therefore,  $F'(L \cup T \cup m) = \sum_\alpha d(V_\alpha) F'(L \cup T \cup m_\alpha)$ , where  $m_\alpha$  is the meridian  $m$  colored by  $\alpha$ . Let  $T_\alpha = T \cup m_\alpha$ . Since  $L$  and  $L'$  are both computable presentations of the  $H$ -stabilization  $(M, T_\alpha, \omega)$  then Theorem 3.10 (proved above) implies  $\frac{F'(L \cup T \cup m_\alpha)}{\Delta_+^r \Delta_-^s} = \frac{F'(L' \cup T \cup m_\alpha)}{\Delta_+^{r'} \Delta_-^{s'}}$ . Summing over  $\alpha$  with coefficients  $d(\alpha)$  one gets  $\frac{F'(L \cup T \cup m)}{\Delta_+^r \Delta_-^s} = \frac{F'(L' \cup T \cup m)}{\Delta_+^{r'} \Delta_-^{s'}}$  where  $m$  is colored by the Kirby color  $\Omega_{g_0}$ . Now in the graph of  $L' \cup T \cup m$ , the meridian  $m$  bounds a disc intersecting some components of  $L'$ . Then applying Lemma 4.10 to the resulting diagram one can eliminate  $m$  and the invariant gets multiplied by  $\Delta_+$ . Thus  $\frac{F'(T \cup L)}{\Delta_+^r \Delta_-^s} = \frac{F'(T \cup L')}{\Delta_+^{r'} \Delta_-^{s'}}$ . If the move  $s_k$  is a blow-down move eliminating  $L_0^0$  then either  $m$  is left linked with some other components (in which case one can apply the above argument) or before the move  $m \cup L_0^0$  was unlinked from the rest of  $L$  and thus both the invariant before and after  $s_k$  are zero. 3.7

## 5. PROOFS OF THE THEOREMS OF SECTION 1

**5.1. A quantization of  $\mathfrak{sl}(2)$  and its associated ribbon category.** In this subsection we consider a ribbon category which underlies the combinatorial invariants defined in Section 1. Fix a positive integer  $r$  and let  $q = e^{\frac{\pi\sqrt{-1}}{r}}$  be a  $2r^{th}$ -root of unity. We use the notation  $q^x = e^{\frac{\pi\sqrt{-1}x}{r}}$ . Here we give a slightly generalized version of quantum  $\mathfrak{sl}(2)$ . Let  $U_q^H \mathfrak{sl}(2)$  be the  $\mathbb{C}(q)$ -algebra given by generators  $E, F, K, K^{-1}, H$  and relations:

$$\begin{aligned} HK &= KH, & HK^{-1} &= K^{-1}H, & [H, E] &= 2E, & [H, F] &= -2F, \\ KK^{-1} &= K^{-1}K = 1, & KEK^{-1} &= q^2E, & KFK^{-1} &= q^{-2}F, & [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

The algebra  $U_q^H \mathfrak{sl}(2)$  is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\ \Delta(H) &= H \otimes 1 + 1 \otimes H, & \varepsilon(H) &= 0, & S(H) &= -H, \\ \Delta(K) &= K \otimes K & \varepsilon(K) &= 1, & S(K) &= K^{-1}, \\ \Delta(K^{-1}) &= K^{-1} \otimes K^{-1} & \varepsilon(K^{-1}) &= 1, & S(K^{-1}) &= K. \end{aligned}$$

Define  $\bar{U}_q^H \mathfrak{sl}(2)$  to be the Hopf algebra  $U_q^H \mathfrak{sl}(2)$  modulo the relations  $E^r = F^r = 0$ .

Let  $V$  be a finite dimensional  $\bar{U}_q^H \mathfrak{sl}(2)$ -module. An eigenvalue  $\lambda \in \mathbb{C}$  of the operator  $H : V \rightarrow V$  is called a *weight* of  $V$  and the associated eigenspace is called a *weight space*. We call  $V$  a *weight module* if  $V$  splits as a direct sum of weight spaces and  $q^H = K$  as operators on  $V$ . Let  $\mathcal{C}$  be the tensor Ab-category of finite dimensional weight  $\bar{U}_q^H \mathfrak{sl}(2)$ -modules (here the ground ring is  $\mathbb{C}$ ).

We will now recall that the category  $\mathcal{C}$  is a ribbon Ab-category. Recall the notation of Section 1. Let  $V$  and  $W$  be objects of  $\mathcal{C}$ . Let  $\{v_i\}$  be a basis of  $V$  and  $\{v_i^*\}$  be a dual basis of  $V^*$ . Then

$$b_V : \mathbb{C} \rightarrow V \otimes V^*, \text{ given by } 1 \mapsto \sum v_i \otimes v_i^* \quad d_V : V^* \otimes V \rightarrow \mathbb{C}, \text{ given by } f \otimes w \mapsto f(w)$$

are duality morphisms of  $\mathcal{C}$ . In [30] Ohtsuki defines an  $R$ -matrix operator defined on  $V \otimes W$  by

$$(24) \quad R = q^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n.$$

where  $q^{H \otimes H/2}$  is the operator given by

$$q^{H \otimes H/2}(v \otimes v') = q^{\lambda \lambda' / 2} v \otimes v'$$

for weight vectors  $v$  and  $v'$  of weights of  $\lambda$  and  $\lambda'$ . Thus, the action of  $R$  on the tensor product of two objects of  $\mathcal{C}$  is well defined and induces an endomorphism on such a tensor product. Moreover,  $R$  gives rise to a braiding  $c_{V,W} : V \otimes W \rightarrow W \otimes V$  on  $\mathcal{C}$  defined by  $v \otimes w \mapsto \tau(R(v \otimes w))$  where  $\tau$  is the permutation  $x \otimes y \mapsto y \otimes x$  (see [26, 30]). Also, in [30] Ohtsuki defines an operator  $\theta$  given by

$$(25) \quad \theta = K^{r-1} q^{-H^2/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} S(F^n) E^n$$

where  $q^{-H^2/2}$  is an operator defined by on a weight vector  $v_\lambda$  by  $q^{-H^2/2}.v_\lambda = q^{-\lambda^2/2} v_\lambda$ . The twist  $\theta_V : V \rightarrow V$  in  $\mathcal{C}$  is defined by  $v \mapsto \theta^{-1} v$  (see [26, 30]).

Remark that the ribbon structure on  $\mathcal{C}$  induce right duality morphisms

$$(26) \quad d'_V = d_V c_{V,V^*}(\theta_V \otimes \text{Id}_{V^*}) \text{ and } b'_V = (\text{Id}_{V^*} \otimes \theta_V) c_{V,V^*} b_V$$

which are compatible with the left duality morphisms  $\{b_V\}_V$  and  $\{d_V\}_V$ .

**5.2. The invariant of oriented trivalent framed graphs  $\mathbb{N}$  through  $U_q^H \mathfrak{sl}(2)$ .** In this subsection show that the categories define in the previous subsection gives rise to an invariant of ribbon graphs which recovers the invariants of trivalent graphs defined in Subsection 1.2. We say a simple weight module is *typical* if its highest weight minus  $(r-1)$  is in the set  $(\mathbb{C} \setminus \mathbb{Z}) \cup \{kr : k \in \mathbb{Z}\} = \mathbb{C} \setminus X_r$ , otherwise we say it is *atypical*. A typical module is  $r$  dimensional. For  $\alpha \in \mathbb{C} \setminus X_r$ , we denote  $V_\alpha$  by the simple weight module with highest weight  $\alpha + r - 1$ .

Let  $F$  be the usual ribbon functor from  $\text{Rib}_{\mathcal{C}}$  to  $\mathcal{C}$ . Let  $\mathbf{A}$  be the set of typical modules. Let  $\mathbf{d} : \mathbf{A} \rightarrow \mathbb{C}$  given by  $\mathbf{d}(V_\alpha) = \mathbf{d}(\alpha)$  where  $\mathbf{d}(\alpha)$  is defined in Equation (2). In [15], it is shown that map  $F' : \{\mathbf{A}\text{-graphs}\} \rightarrow \mathbb{C}$  given by Equation (11) is a well defined invariant. In particular,  $(\mathbf{A}, \mathbf{d})$  is an ambidextrous pair.

Next we will show that  $F'$  can be used to define  $\mathbb{N}$ . In particular, we will recall how  $F'$  extends to an invariant of trivalent framed graphs whose edges are colored by element of  $\mathbb{C} \setminus X_r$  (for more details see [16]). This extension requires the choice of a certain family of morphisms in  $\mathcal{C}$ . Such a family is given in [13] when  $r$  is odd. Here we will show that another family can be deduced from the computation of [6] for any  $r \geq 2$ .

Let  $U$  be the quantization of quantum  $\mathfrak{sl}(2)$  considered in [6]. The algebras  $U$  and  $U_q^H \mathfrak{sl}(2)$  have the same underlying structure. However, they differ in two main ways: 1) the element  $K$  in  $U$  should be considered the square root of the corresponding element  $K$  in  $U_q^H \mathfrak{sl}(2)$ , 2)  $U_q^H \mathfrak{sl}(2)$  has the additional generator  $H$ . The first difference essentially has no effect on the corresponding topological invariants. As explained in Subsection 5.1, the generator  $H$  allows the category  $\mathcal{C}$  to be braided.

Let  $U\text{-cat}$  be the category of weight  $U$ -modules. Consider the functor  $\mathcal{C} \rightarrow U\text{-cat}$  which is the identity at the level of vector spaces and linear maps and sends a weight  $\overline{U}_q^H \mathfrak{sl}(2)$ -module to the weight  $U$ -module determined by the action of the generators  $K', E', F'$  of  $U$  given by  $q^{H/2}, q^{-H/2}E, Fq^{H/2}$ , respectively. This functor sends  $V_\alpha$  to  $V^a$  where  $V^a$  is the highest weight  $U$ -module of highest weight  $a = (\alpha + r - 1)/2$  considered in [6]. Let  $\cap_{a,r-1-a} : V^a \otimes V^{r-1-a} \rightarrow \mathbb{C}$  be the map defined in Equation (1.2) of [6]. Let  $\cap_{\alpha,-\alpha} : V_\alpha \otimes V_{-\alpha} \rightarrow \mathbb{C}$  be the corresponding morphism in  $\mathcal{C}$ . For  $\alpha \in \mathbb{C} \setminus X_r$  define the isomorphism  $w_\alpha : V_\alpha \rightarrow V_{-\alpha}^*$  by

$$w_\alpha = (\cap_{\alpha,-\alpha} \otimes \text{Id}_{V_{-\alpha}^*}) \circ (\text{Id}_{V_\alpha} \otimes b_{V_{-\alpha}}).$$

**Proposition 5.1.** *The family  $\{w_\alpha\}_{\alpha \in \mathbb{C} \setminus X_r}$  satisfy*

$$(27) \quad d_{V_\alpha} \circ (w_{-\alpha} \otimes \text{Id}_{V_\alpha}) = d'_{V_{-\alpha}} \circ (\text{Id}_{V_{-\alpha}} \otimes w_\alpha) \in \text{Hom}_{\mathcal{C}}(V_{-\alpha} \otimes V_\alpha, \mathbb{C})$$

for all  $\alpha \in \mathbb{C} \setminus X_r$ .

*Proof.* Using the naturality of the braiding and the formula for  $d'_{V_{-\alpha}}$  given in Equation (26) we can rewrite Equation (27) as

$$\cap_{-\alpha,\alpha} = \cap_{\alpha,-\alpha} c_{V_{-\alpha}, V_\alpha} (\theta_{V_{-\alpha}} \otimes \text{Id}_{V_\alpha}).$$

This equation is equivalent to the following equation in  $U\text{-mod}$ :

$$\cap_{b,a} = q^{2ab} \cap_{a,b} \circ \binom{a}{b} R$$

where  $b = r - 1 - a$  and  $\binom{a}{b} R$  is the map defined in Equation (1.4) of [6]. The last equality is an easy consequence of [6, Proposition 3.1]. 5.1

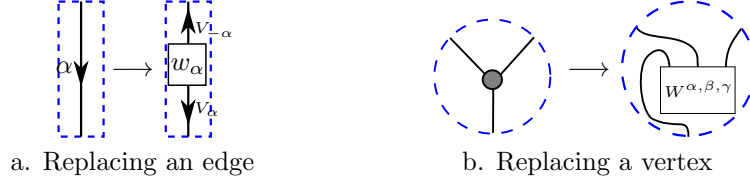
Consider the map  $Y_c^{a,b} : V^c \rightarrow V^a \otimes V^b$  define in Theorem 1.7 of [6] where  $a, b, c \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$  and  $a + b - c \in \{0, 1, \dots, r - 1\}$ . Using the functor above the map  $Y_c^{a,b}$  corresponds to a non-zero morphism  $Y_{-\gamma}^{\alpha,\beta} : V_{-\gamma} \rightarrow V_\alpha \otimes V_\beta$  in  $\mathcal{C}$  where  $\alpha, \beta, \gamma \in \mathbb{C} \setminus X_r$  with  $\alpha + \beta + \gamma \in H_r$ . Define  $W^{\alpha,\beta,\gamma} : \mathbb{C} \rightarrow V_\alpha \otimes V_\beta \otimes V_\gamma$  as the morphism  $W^{\alpha,\beta,\gamma} = (Y_{-\gamma}^{\alpha,\beta} \otimes w_\gamma^{-1}) b_{V_{-\gamma}}$ . Now Lemma 1.8 of [6] implies that

$$(28) \quad (d_{V_\alpha} \otimes \text{Id}_{V_\beta \otimes V_\gamma \otimes V_\alpha}) \circ (\text{Id}_{V_\alpha^*} \otimes W^{\alpha,\beta,\gamma} \otimes \text{Id}_{V_\alpha}) \circ b'_{V_\alpha} = W^{\beta,\gamma,\alpha}.$$

Since the family  $\{V_\delta, w_\delta\}_{\delta \in \mathbb{C} \setminus X_r}$  satisfies Equation (27) then in the terminology of [16] it is called *basic data*. Moreover, with this basic data the pair  $(\mathbb{C} \setminus X_r, \mathbf{d})$  is *trivalent-ambidextrous* (see Definition 1 of [16]). Thus, as explained after Lemma 2 in [16] the invariant  $F'$  extends to an invariant of oriented trivalent framed graphs whose edges are colored by elements of the set  $\mathbb{C} \setminus X_r$ . This extension can be summarized as follows.

Let  $\Gamma$  be an oriented trivalent framed graph in  $S^3$  and  $\Sigma$  be a thickening of  $\Gamma$  to a surface using the framing. We construct a framed  $\mathcal{C}$ -colored ribbon graph with coupons  $\Gamma'$  embedded in  $\Sigma$ . First we decompose  $\Sigma$  in bands, discs and annuli corresponding respectively to the edges, 3-valent vertices and loops of  $\Gamma$ . Then we replace a band colored by  $\alpha$  by two edges colored by  $V_\alpha$  and  $V_{-\alpha}$  and a bivalent coupon filled with  $w_\alpha$  as shown in Figure 5a. Then we put in each disk a coupon filled with one of the morphisms  $W^{\alpha,\beta,\gamma}$  as shown in Figure 5b. The color of an  $\alpha$ -colored loop is replaced by  $V_\alpha$ . The construction of  $\Gamma'$  involve some choice but the quantity  $F'(\Gamma')$  does not depend of these choices. Define  $\mathbf{N}(\Gamma) = F'(\Gamma')$ .

By definition the extension of  $F'$  can be computed using the formulas of [6] via the functor  $\mathcal{C} \rightarrow U\text{-mod}$  discussed above. In particular, the relations of Section 1.2 are then consequences of computations given in [6] and of properties of  $F'$ . Let us be more precise. Suppose  $T$  and  $T'$  are two 1-1  $\mathcal{C}$ -colored ribbon graphs whose colors belongs to  $\mathbf{A}$  such that  $F(T) = F(T')$  then Proposition 4.4 implies  $F'(\text{tr}(T)) = F'(\text{tr}(T'))$  where  $\text{tr}$  is the braid closure of the tangle. An analogous relation between  $F$  and  $\mathbf{N}$  exists. We now explain why the axioms of Section 1.2 hold.

FIGURE 5. Construction of a  $\mathcal{C}$ -colored ribbon graph with a trivalent graph.

The invariant  $F$  satisfies a property analogous to Axiom (N a) and so this axiom is a direct consequence of this fact. The endomorphism set  $\text{Hom}_{\mathcal{C}}(\mathbb{C}, V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma})$  is zero if  $\alpha + \beta + \gamma \notin H_r$  and so Axiom (N b) follows.

The endomorphism represented by  $T$  in Axiom (N c) is in  $\text{Hom}_{\mathcal{C}}(V_{\alpha}, V_{\beta})$  which is equal to  $\{0\}$  if  $\alpha \neq \beta$  and  $\mathbb{C} \cdot \text{Id}_{V_{\alpha}}$  if  $\alpha = \beta$ . In the latter case,  $F(T)$  is a scalar times  $\text{Id}_{V_{\alpha}}$  and this scalar is by definition  $d(\alpha)^{-1}$  times  $N(\hat{T})$  where  $\hat{T}$  is the closure of  $T$  which appears in the right hand side of the axiom. Replacing  $T$  by this scalar times a strand representing the identity of  $V_{\alpha}$ , we get the equality in (N c).

The first identity of (N d) follows directly from the definition of  $F'$ . Lemma 1.10 of [6] implies  $N(\bigcirc) = 1$  and thus proves the second equalities in Axiom (N d). The last equality follows from the fact that  $\text{Hom}_{\mathcal{C}}(V_{\alpha}, \mathbb{C}) = \{0\}$ .

The invariant  $F$  vanishes on any closed graph  $T$  colored by modules in the set  $\{V_{\alpha}\}_{\alpha \in \mathbb{C} \setminus X_r}$ . This implies that  $F'$  and  $N$  vanish on any split graph  $T \sqcup T'$  where both  $T$  and  $T'$  are colored by the modules in the set  $\{V_{\alpha}\}_{\alpha \in \mathbb{C} \setminus X_r}$  and thus Property (6) in the list of axioms holds.

The space  $\text{Hom}_{\mathcal{C}}(\mathbb{C}, V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma})$  is one dimensional and generated by  $W^{\alpha, \beta, \gamma}$  (assuming that  $\alpha + \beta + \gamma \in H_r$ ). Hence the morphisms represented by  $T$  and  $T'$  in (N e) are equal to  $\lambda \cdot W^{\alpha, \beta, \gamma}$  and  $\lambda' \cdot W^{-\alpha, -\beta, -\gamma}$  respectively. The scalars  $\lambda$  and  $\lambda'$  are the factors of the right hand side because  $N$  is 1 on the  $\Theta$  graphs.

Equation (3.1) of [6] implies Axioms (N f) and (N g). A computation of  $F'$  of the Hopf link can be found in [15]. Equivalently, Axiom (N h) can be deduced from the other axioms.

Axiom (N i) follows from the decomposition of modules  $V_{\alpha} \otimes V_{\beta} \simeq \bigoplus_{k \in H_r} V_{\alpha + \beta + k}$ . Here we use that if  $\gamma = \alpha + \beta + k$ , then  $\text{Hom}_{\mathcal{C}}(V_{\gamma}, V_{\alpha} \otimes V_{\beta})$  is generated by the element corresponding to  $W^{\alpha, \beta, -\gamma}$  through the isomorphism  $\text{Hom}_{\mathcal{C}}(V_{\gamma}, V_{\alpha} \otimes V_{\beta}) \cong \text{Hom}_{\mathcal{C}}(\mathbb{C}, V_{\alpha} \otimes V_{\beta} \otimes V_{-\gamma})$  and similarly for  $\text{Hom}_{\mathcal{C}}(V_{\alpha} \otimes V_{\beta}, V_{\gamma}) \cong \text{Hom}_{\mathcal{C}}(\mathbb{C}, V_{-\beta} \otimes V_{-\alpha} \otimes V_{\gamma})$ . Finally, the  $6j$ -symbols computed in [6] are by definition the coefficients that appear in (N j).

**5.3. The relative  $G$ -modular structure on  $\mathcal{C}$ .** Here we show that the categories considered earlier in this section are relative  $G$ -modular categories. Let  $G$  be the additive group  $\mathbb{C}/2\mathbb{Z}$ ,  $\bar{X} = \mathbb{Z}/2\mathbb{Z} \subset G$  and  $\mathbb{Z} = \mathbb{Z}$  (here we use additive notation). We will now show that  $\mathcal{C}$  is a relative  $G$ -modular category relative to  $\bar{X}$  with modified dimension  $d$  and periodicity group  $\mathbb{Z}$ . To do this we will show that Conditions (1)–(8) of Definition 3.2 hold.

For  $g \in G$ , define  $\mathcal{C}_g$  as the full sub-category of weight modules with weights congruent to  $g$  mod 2. Then it is easy to see that  $\{\mathcal{C}_g\}_{g \in G}$  is a  $G$ -grading in  $\mathcal{C}$ . Moreover,  $\bar{X}^{-1} = \bar{X}$  and  $G$  can not be covered by a finite number of translated copies of  $\bar{X}$ . Thus, Conditions (1) and (4) are satisfied.

Recall that  $\mathbf{A}$  is the set of typical modules and  $d : \mathbf{A} \rightarrow \mathbb{C}$  is the function given by  $d(V_{\alpha}) = d(\alpha)$  where  $d(\alpha)$  is defined in Equation (2). Also, in [15] it is shown that  $(\mathbf{A}, d)$  is an ambidextrous pair. Thus, if  $g \in G \setminus \bar{X}$  then by definition the simple modules of  $\mathcal{C}_g$  are all typical and Condition

(5) holds. Moreover, it follows that the category  $\mathcal{C}_g$  is semi-simple if  $g \in G \setminus \overline{X}$  (see Lemma 6.1 where we prove a general statement).

For  $t \in \mathbb{Z}$ , let  $\varepsilon^t$  be the one dimensional vector space  $\mathbb{C}$  endowed with the  $\overline{U}_q^H \mathfrak{sl}(2)$ -action determined by

$$Ev = Fv = 0, \quad Kv = v, \quad Hv = 2rtv$$

for any  $v \in \varepsilon^t$ . Then  $\varepsilon^t$  is a weight module in  $\mathcal{C}_0$ . Since, the action of  $E$  and  $F$  on  $\varepsilon^t$  is zero, it is easy to see that  $\{\varepsilon^t\}_{t \in \mathbb{Z}}$  is commutative set of objects in  $\mathcal{C}$  and  $\varepsilon^t \otimes \varepsilon^{t'} = \varepsilon^{t+t'}$ . Moreover,

$$(29) \quad \varepsilon^t \otimes V_\alpha = V_\alpha \otimes \varepsilon^t = V_{\alpha+2rt}$$

and it follows that  $\{\varepsilon^t\}_{t \in \mathbb{Z}}$  is a free realization of  $\mathbb{Z}$  in  $\mathcal{C}$ , i.e. Condition (2) holds. For  $g \in G$ , the simple modules of  $\mathcal{C}_g$  are all the typical modules  $V_\alpha$  such that  $\alpha + r - 1 \equiv g \pmod{2}$ . Equation (29) implies this set of typical modules is the union the simple  $\mathbb{Z}$ -orbits  $\tilde{V}_{\alpha+i}$  where  $i$  runs over the set  $\{0, 1, \dots, 2r-1\}$  and  $\alpha$  is a complex number such that  $\alpha \equiv g \pmod{2}$ . Therefore, Condition (6) holds.

Let  $G \times \mathbb{Z} \rightarrow \mathbb{C}^*$  be the map given by  $(g, t) \mapsto q^{2rt\alpha}$  where  $\alpha$  is any complex number such that  $\alpha + r - 1 \equiv g \pmod{2}$ . Note this map is well define since  $q$  is a  $2r$ th root of unity. If  $V$  is a weight module then

$$(30) \quad c_{V, \varepsilon^t} = \tau \circ (K^{rt} \otimes \text{Id}) \quad \text{and} \quad c_{\varepsilon^t, V} = \tau \circ (\text{Id} \otimes K^{rt})$$

where  $\tau$  is the flip map  $x \otimes y \mapsto y \otimes x$ . Thus, Condition (3) holds.

Finally, the computations given at the end of Subsection 1.2 shows that Condition (7) holds. A similar computation is given in [15] to show that  $F(H(V, W)) \neq 0$  where  $H(V, W)$  is the long Hopf link whose long edge is colored by an object  $V \in \mathcal{A}$  and whose circle component is colored an object  $W \in \mathcal{A}$ . Thus, Condition (8) holds.

**5.4. The 3-manifold invariant  $\mathbf{N}$ .** In this subsection we prove Theorems 1.4, 1.7 and 1.8 and Proposition 1.5. In Section 1 we only considered compatible triples  $(M, T, \omega)$  where  $T$  was a framed trivalent graph in a 3-manifold  $M$  whose edges were colored by elements of  $\mathbb{C} \setminus X_r$ . We only consider such triples  $(M, T, \omega)$  in this section. If  $T \neq \emptyset$  then by definition  $(M, T, \omega)$  is  $T$ -admissible. Also, if  $(M, T, \omega)$  is computable as defined in Section 1 then  $(M, T, \omega)$  is computable as defined in Section 3.4.

In the last section we showed that  $\mathcal{C}$  was a relative  $G$ -modular category. Therefore, the general theory of Subsection 3.4 imply the existence of 3-manifold invariants  $\mathbf{N}$  and  $\mathbf{N}^0$ . The main results of Section 1 follow from this general theory and the fact that  $\mathbf{N}$  is defined as an extension of  $F'$ . In particular,

Theorem 3.7 implies Theorem 1.4

Theorem 3.10 implies Theorem 1.7

Theorem 3.12 implies Theorem 1.8

Note here that if  $H = H(\alpha, \beta)$  is the long Hopf-link in Theorem 1.7 then  $\langle H \rangle = F'(H)/d(\beta) = (-1)^{r-1}rq^{\alpha\beta}/d(\beta)$ .

*Proof of Proposition 1.5.* Let  $(M, T, \omega)$  be a compatible triple and  $L$  be a link which gives rise to a surgery presentation of  $M$ . Then the image of  $\omega \in \text{Hom}(H_1(M \setminus T, \mathbb{Z}), \mathbb{C}/2\mathbb{Z})$  is generated by the values of  $\omega$  on the meridian of  $L \cup T$ . As  $\omega$  is not integral, its image is not contained in  $\mathbb{Z}/2\mathbb{Z}$ , and  $L \cup T$  has an edge  $e$  with  $g_\omega(e) \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$ .

Suppose  $L_i$  is a component of  $L$  such that  $g_\omega(L_i) \in \overline{X} = \mathbb{Z}/2\mathbb{Z}$ . If we slide the component  $e$  over  $L_i$ , after this sliding the  $G$ -color of  $L_i$  is  $g_\omega(e)g_\omega(L_i)$  which is not in  $\overline{X}$  (this is imposed

1.5

**Remark 5.2.**  $N^0(M, T, \omega)$  can be extended by allowing the tangle part  $T$  to contain any  $\mathcal{C}$ -colored link. In particular, let  $L^J$  denotes the coloring of a link by the two dimensional representation with highest weight 1. Then  $N^0(S^3, L^J, 0)$  is just the Kauffman bracket of  $L$  evaluated at  $A^2 = q$  and for a 3-manifold,  $N^0(M, L^J, 0)$  is a generalization of it for links in 3-manifolds.

### 5.5. Proof of Proposition 2.5.

*Proof of Proposition 2.5.* Let  $W = V_\alpha \otimes V_\beta \otimes V_\gamma$  and  $f_a = F \left( \Omega_a^{\text{circle}} \right) \in \text{End}_{U_q(\mathfrak{sl}_2)}(W)$ . We will show the image of  $f_a$  is the trivial module. By Lemma 4.9 we can do handle-slides, blow-up moves and blow-down moves on the circle component of the graph representing  $f_a \otimes \text{Id}_{V_0}$  to obtain via Proposition 4.4 the equality of morphisms:

$$(31) \quad c_{W,V_0} \circ (f_a \otimes Id_{V_0}) = c_{V_0,W}^{-1} \circ (f_a \otimes Id_{V_0})$$

where the braidings  $c_{W,V_0}, c_{V_0,W}^{-1} : W \otimes V_0 \rightarrow V_0 \otimes W$  are given by

$$c_{W,V_0} = \tau \circ R = \tau \circ q^{H \otimes H/2} (\text{Id} \otimes \text{Id} + (q - q^{-1})E \otimes F + \cdots)$$

and

$$c_{V_0, W}^{-1} = R^{-1} \circ \tau = (\text{Id} \otimes \text{Id} + (-q + q^{-1})E \otimes F + \dots)q^{-H \otimes H/2} \circ \tau.$$

Here the dots “...” are linear combination of power  $(E \otimes F)^k$  with  $k \geq 2$ .

Let  $x \in V_\alpha \otimes V_\beta \otimes V_\gamma$  and set  $y = f_\alpha(x)$ . The module  $V_0$  has its weights in  $H_r$  and, as  $r$  is odd,  $V_0$  has a non zero weight vector  $v_0$  of weight 0. The vectors  $\{E^k.v_0, F^k.v_0 : k = 1 \dots \frac{r-1}{2}\}$  form with  $v_0$  a basis of  $V_0$ . Let  $V'$  be the vector space generated by  $\{E^k.v_0, F^k.v_0 : k = 2 \dots \frac{r-1}{2}\}$ . Applying (31) to  $x \otimes v_0$  we have

$$c_{W,V_0}(y \otimes v_0) = c_{V_0,W}^{-1}(y \otimes v_0)$$

with

$$(32) \quad c_{W,V_0}(y \otimes v_0) = \tau \circ q^{H \otimes H/2}(y \otimes v_0 + (q - q^{-1})Ey \otimes Fv_0 + Y_1')$$

$$(33) \quad = v_0 \otimes y + (q - q^{-1})Fv_0 \otimes K^{-1}Ey + Y_2'$$

where  $Y'_1 \in W \otimes V'$  and  $Y'_2 \in V' \otimes W$ . The last equality comes from the facts that  $Hv_0 = 0$  and  $HFv_0 = -2Fv_0$ . Similarly,

$$(34) \quad c_{V_0 W}^{-1}(y \otimes v_0) = v_0 \otimes y - (q - q^{-1})Ev_0 \otimes Fy + Y'_3$$

where  $Y'_3 \in V' \otimes W$ . Setting the above equations equal we have  $K^{-1}Ey = Fy = 0$ . So  $Ey = 0$  and also,  $(q - q^{-1})(EF - FE)y = (K - K^{-1})y = 0$ . Thus,  $K^2y = y$ ,  $Ey = 0$  and  $Fy = 0$  and this holds for any choice of  $a$  and of  $x$ ; but, since  $K$  acts as  $q^H$  and the weights of  $W$  are in  $2\mathbb{Z}$ , we have that the eigenvalues of  $K$  are in  $q^{2\mathbb{Z}} \not\ni -1$ . Thus  $Ky = y$  and  $f_a(x)$  is an invariant vector of  $W$ .

Then as  $\text{Hom}_{U_q(sl_2)}(\mathbb{C}, V_\alpha \otimes V_\beta \otimes V_\gamma) \simeq \text{Hom}_{U_q(sl_2)}(V_\alpha \otimes V_\beta \otimes V_\gamma, \mathbb{C}) \simeq \mathbb{C}$ , the left hand side of Equation (7) has to be proportional to the right hand side. Let  $\lambda \in \mathbb{C}$  be the coefficient of proportionality. To compute  $\lambda$ , we consider the value by  $\mathbf{N}$  of the braid closure of the graphs in this equality. The braid closure of the right side is  $\lambda$  times the  $\Theta$ -graph on which  $\mathbf{N}$  has value 1.

Thus  $\lambda$  is equal to  $N$  of the braid closure of the graph in the left hand side. But this graph is a connected sum of 3 Hopf links and its value is thus given by

$$\lambda = \sum_{k \in H_r} d(a+k)^{-1} r^3 q^{(a+k)\alpha} q^{(a+k)\beta} q^{(a+k)\gamma} = \sum_{k \in H_r} \frac{r^3}{d(a+k)}.$$

But for any  $b \in \mathbb{C} \setminus X_r$ , we have  $d(b)^{-1} = r^{-1} \sum_{l \in H_r} q^{lb}$  thus

$$\lambda = r^2 \sum_{k \in H_r} \sum_{l \in H_r} q^{l(a+k)} = r^2 \sum_{l \in H_r} q^{la} \sum_{k \in H_r} q^{lk}$$

Now if  $l \in H_r \setminus \{0\}$ , then  $\sum_{k \in H_r} q^{lk} = 0$  thus only the term for  $l = 0 \in H_r$  contributes and  $\lambda = r^3$ . 2.5

## 6. THE OTHER QUANTUM GROUPS

In this section we recall the results of [14] and show that they imply the existence of relative  $G$ -modular categories associated with the quantum group of any simple Lie algebra.

Let  $\mathfrak{g}$  be a simple finite-dimensional complex Lie algebra of rank  $n$  and dimension  $2N + n$  with a root system. Fix a set of simple roots  $\{\alpha_1, \dots, \alpha_n\}$  and let  $R^+$  be the corresponding set of positive roots. Also, let  $A = (a_{ij})_{1 \leq i, j \leq n}$  be the Cartan matrix corresponding to these simple roots. There exists a diagonal matrix  $D = \text{diag}(d_1, \dots, d_n)$  such that  $DA$  is symmetric and positive definite and  $\min\{d_i\} = 1$ . Let  $\mathfrak{h}$  be the Cartan subalgebra of  $\mathfrak{g}$  generated by the vectors  $H_1, \dots, H_n$  where  $H_j$  is determined by  $\alpha_i(H_j) = a_{ji}$ . Let  $L_R$  be the root lattice which is the  $\mathbb{Z}$ -lattice generated by the simple roots  $\{\alpha_i\}$ . Let  $\langle \cdot, \cdot \rangle$  be the form on  $L_R$  given by  $\langle \alpha_i, \alpha_j \rangle = d_i a_{ij}$ . Let  $L_W$  be the weight lattice which is the  $\mathbb{Z}$ -lattice generated by the elements of  $\mathfrak{h}^*$  which are dual to the elements  $H_i$ ,  $i = 1 \dots n$ . Let  $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in L_W$ .

Let  $r$  be an odd integer such that  $r \geq 3$  and  $r \notin 3\mathbb{Z}$  if  $\mathfrak{g} = G_2$ . Let  $q = e^{2i\pi/r}$  and for  $i = 1, \dots, n$ , let  $q_i = q^{d_i}$ . For  $x \in \mathbb{C}$  and  $k, l \in \mathbb{N}$  we use the notation:

$$q^x = e^{\frac{2i\pi x}{r}}, \quad \{x\}_q = q^x - q^{-x}, \quad [x]_q = \frac{\{x\}_q}{\{1\}_q}, \quad [k]_q! = [1]_q [2]_q \cdots [k]_q, \quad \begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[l]_q!}.$$

Remark for  $x \in \mathbb{C}$ ,  $\{x\} = 0$  if and only if  $x \in \frac{r}{2}\mathbb{Z}$ .

The *unrolled quantum group*  $\mathcal{U}^H$  is the algebra generated by  $K^\beta, X_i, X_{-i}, H_i$  for  $\beta \in L_W$ ,  $i = 1, \dots, n$  with relations

$$(35) \quad K^0 = 1, \quad K^\beta K^\gamma = K^{\beta+\gamma}, \quad K^\beta X_{\sigma i} K^{-\beta} = q^{\sigma \langle \beta, \alpha_i \rangle} X_{\sigma i},$$

$$(36) \quad [X_i, X_{-j}] = \delta_{ij} \frac{K^{\alpha_i} - K^{-\alpha_i}}{q_i - q_i^{-1}},$$

$$(37) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} X_{\sigma i}^k X_{\sigma j} X_{\sigma i}^{1-a_{ij}-k} = 0, \text{ if } i \neq j$$

$$(38) \quad [H_i, X_{\epsilon j}] = \sigma a_{ij} X_{\sigma j}, \quad [H_i, H_j] = [H_i, K^\beta] = 0$$

where  $\sigma = \pm 1$ .

The algebra  $\mathcal{U}^H$  is a Hopf algebra with coproduct  $\Delta$ , counit  $\epsilon$  and antipode  $S$  defined by

$$\Delta(X_i) = 1 \otimes X_i + X_i \otimes K^{\alpha_i}, \quad \Delta(X_{-i}) = K^{-\alpha_i} \otimes X_{-i} + X_{-i} \otimes 1,$$

$$\Delta(K^\beta) = K^\beta \otimes K^\beta, \quad \epsilon(X_i) = \epsilon(X_{-i}) = 0, \quad \epsilon(K^{\alpha_i}) = 1,$$

$$S(X_i) = -X_i K^{-\alpha_i}, \quad S(X_{-i}) = -K^{\alpha_i} X_{-i}, \quad S(K^\beta) = K^{-\beta},$$

$$\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1, \quad \epsilon(H_i) = 0, \quad S(H_i) = -H_i.$$

$\mathcal{U}^H$  has an Hopf ideal  $I$  which contains the  $r^{\text{th}}$  powers of the roots vectors (see [14]).

Also in [14], a full subcategory  $\mathcal{D}^\theta$  of the category of finite dimensional representations of  $\mathcal{U}^H$  is shown to be ribbon. Let us describe briefly its modules.

A weight vector of weight  $\lambda \in \mathfrak{h}^*$  in a  $\mathcal{U}^H$ -module is a vector on which  $H_i$  acts by  $\lambda(H_i)$ . A weight vector  $v$  is an highest weight vector if  $E_i.v = 0, \forall i = 1 \cdots n$ . A weight module is a  $\mathcal{U}^H$ -module which satisfy:

- (1) it is finite dimensional over  $\mathbb{C}$ ,
- (2) it has a base of weight vectors,
- (3) the elements  $K^{\sum_i \lambda_i \alpha_i}$  act on it as  $q^{\sum_i \lambda_i H_i}$ ,
- (4) elements of  $I$  vanish on it.

Any weight module  $V$  has an highest weight vector and it is unique (up to a scalar) if  $V$  is irreducible. Moreover the set of isomorphic classes of irreducible weight modules is in bijection with  $\mathfrak{h}^*$ . We will write  $V_\lambda$  for an irreducible module with highest weight  $\lambda + (r-1)\rho$ .

$\mathcal{D}^\theta$  is a full sub-tensor category of the category  $\mathcal{D}$  of weight modules (conjecturally,  $\mathcal{D}^\theta = \mathcal{D}$ ). The category  $\mathcal{D}$  (and also  $\mathcal{D}^\theta$ ) is  $G$ -graded where  $G = \mathfrak{h}^*/L_R \cong (\mathbb{C}^*)^n$ . The weights of a module in  $\mathcal{D}_g$  are all the same modulo  $L_R$ . For any  $\lambda \in L_W$ ,  $K^{r\lambda}$  acts as the same scalar denoted  $g(K^{r\lambda})$  on any module of  $\mathcal{D}_g$ .

Let  $\mathbf{A}$  be the family of irreducible weight modules with highest weight  $\lambda$  such that  $q^{2\langle \lambda + \rho, \beta \rangle + m\langle \beta, \beta \rangle} \neq 1$  for all  $\beta \in R^+$  and  $m \in \{0, \dots, r-1\}$ . Modules in  $\mathbf{A}$  are called typical, they are the  $r^N$ -dimensional simple modules, they all belongs to  $\mathcal{D}^\theta$  and their categorical dimension vanishes.

**Lemma 6.1.** *Typical modules are the simple projective modules of  $\mathcal{D}$ . Hence the subcategory  $\mathcal{D}_g$  is semi-simple iff all its simple modules are typical.*

*Proof.* Every simple module  $V_\lambda \in \mathcal{D}_g$  is a quotient of a  $r^N$ -dimensional module  $\overline{M}(\lambda)$  (which are the “Verma module” for  $\mathcal{D}$  see [9, Section 3.1]) generated by an highest weight vector of weight  $\lambda$ . In particular, if  $V_\lambda$  is not typical, then  $\overline{M}(\lambda)$  is not semi-simple and the category  $\mathcal{C}_g \ni V_\lambda$  is not semi-simple.

Let  $N^+$  (resp  $N^-$ ) be the subalgebra of  $\mathcal{U}^H$  generated by the elements  $X_i$  (resp  $X_{-i}$ ) for  $i = 1 \cdots n$ . Then  $N^+/(I \cap N^+)$  (resp  $N^-/(I \cap N^-)$ ) possess an unique highest weight vector  $x_+$  (resp an unique lowest weight vector  $x_-$ ). As  $\dim_{\mathbb{C}}(N^+/(I \cap N^+)) = \dim_{\mathbb{C}}(N^-/(I \cap N^-)) = r^N$ , we have that for a typical module  $V_\lambda$  with highest weight vector  $v_+$  and lowest weight vector  $v_-$ ,  $x_- \cdot v_+ \in \mathbb{C}^* v_-$  and  $x_+ \cdot v_- \in \mathbb{C}^* v_+$ . Now if  $W \rightarrow V_\lambda$  is an epimorphism, it is surjective and  $v_+$  has a preimage  $w$ . Let  $w_+ = x_+ x_- \cdot w$ . Then  $w_+$  is an highest weight vector of  $W$  (by maximality of  $x_+$ ) which is sent to a non zero multiple of  $v_+$ . The usual property of Verma modules apply to  $\overline{M}(\lambda)$  and there is a unique map  $V_\lambda = \overline{M}(\lambda) \rightarrow V$  which sends the highest weight vector of  $V_\lambda$  to  $w_+$ . This map gives a section of the epimorphism and thus the typical module  $V_\lambda$  is projective. When all simple modules of  $\mathcal{D}_g$  are typical, by an easy induction (considering an irreducible quotient) we get that any finite dimensional module of  $\mathcal{D}_g$  is completely reducible. 6.1

In particular this is true if  $g \notin \overline{X}$  where  $\overline{X} \subset \mathfrak{h}^*/L_R$  is formed by the weights  $\lambda$  such that  $\exists \beta \in R^+$  with  $2\langle \lambda, \beta \rangle \in \mathbb{Z}$ . Remark that for these  $g$ ,  $\mathcal{D}_g \subset \mathcal{D}^\theta$  because  $\mathcal{D}^\theta$  contains all typical modules.

If  $V_\lambda \in \mathbf{A}$  has highest weight  $\lambda + (r-1)\rho$ , let

$$d(V_\lambda) = \prod_{\alpha \in R^+} \frac{\{\langle \lambda, \alpha \rangle\}}{\{r\langle \lambda, \alpha \rangle\}}$$

then  $(\mathbf{A}, d)$  is an ambi pair. The ingredient in [14] to compute  $d$  is the computation of the image by  $F'$  of the Hopf link  $H$  colored by  $V_\lambda, V_\mu$  which is  $F'(H) = q^{2\langle \lambda, \mu \rangle}$ .



For  $t \in \mathbb{Z} = rL_R \cong (r\mathbb{Z})^n$ , let  $\varepsilon^t$  be the vector space  $\mathbb{C}$  endowed with the action of  $\mathcal{U}^H$  given by  $X_{\pm i} = 0$ , and by being a weight space of weight  $t$ . Then  $(\varepsilon^t)_{t \in \mathbb{Z}}$  is a free realization of  $\mathbb{Z}$  in  $\mathcal{D}_1^\theta$  and if  $V \in \mathcal{D}_g^\theta$ , the square of braiding on  $V \otimes \varepsilon^t$  is given by  $g^{\bullet t} = q^{2\langle t, \lambda \rangle}$  for any weight  $\lambda \in V$  (two weights in  $V$  differ by an element of  $L_R$  and  $\langle \mathbb{Z}, L_R \rangle \subset r\mathbb{Z}$ ).

Finally, the twist on  $V_\lambda$  is given by the scalar  $q^{\langle \lambda, \lambda \rangle - (r-1)^2 \langle \rho, \rho \rangle}$  so we have after a computation similar to that of Section 1.2:

$$\begin{aligned} \Delta_+ &= q^{-2(r-1)^2 \langle \rho, \rho \rangle} \sum_{k \in L_R / (rL_R)} q^{\langle k, k+2\rho \rangle} = q^{-3(r-1)^2 \langle \rho, \rho \rangle} \sum_{k \in L_R / (rL_R)} q^{\langle k+(1-r)\rho, k+(1-r)\rho \rangle} \\ \Delta_+ &= q^{-3(r-1)^2 \langle \rho, \rho \rangle} \sum_{k \in L_R / (rL_R)} q^{\langle k, k \rangle}. \end{aligned}$$

Thus at least if  $r$  is coprime with  $\det(a_{i,j})$  then  $|\Delta_+| = r^{\frac{n}{2}} \neq 0$ . In general we have the following theorem

**Theorem 6.2.** *If  $\sum_{k \in L_R / (rL_R)} q^{\langle k, k \rangle} \neq 0$  then  $\mathcal{D}^\theta$  is  $G$ -modular relative to  $\overline{X}$  with modified dimension  $d$  and periodicity group  $\mathbb{Z}$ .*

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INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, RUE RENÉ DESCARTES 7, 67084 STRASBOURG, FRANCE  
*E-mail address:* `costanti@math.unistra.fr`

MATHEMATICS & STATISTICS, UTAH STATE UNIVERSITY, LOGAN, UTAH 84322, USA  
*E-mail address:* `nathan.geer@usu.edu`

UMR 6205, LMBA, UNIVERSITÉ DE BRETAGNE-SUD, UNIVERSITÉ EUROPÉENNE DE BRETAGNE, BP 573, 56017 VANNES, FRANCE  
*E-mail address:* `bertrand.patureau@univ-ubs.fr`